# A GENERALIZATION OF THE OWA OPERATOR TO EVALUATE NON MONOTONIC QUANTIFIERS


**Ludovic Liétard**
IRISA/IUT, rue Edouard Branly,
BP 30219 22302, Lannion cedex France
Ludovic.Lietard@univ-rennes1.fr

**Daniel Rocacher**
IRISA/ENSSAT, rue de Kerampont
BP 447, 22305, Lannion cedex France
rocacher@enssat.fr



**Abstract**

This paper is devoted to the evaluation of quantified statements in flexible querying of relational databases. Its contribution is a new approach based on the arithmetic on fuzzy numbers ($\mathbb{N}_f$, $\mathbb{Z}_f$, $\mathbb{Q}_f$) expressing well-known but gradual numbers. The advantage of this proposition is that no particular assumption is made on the monotonicity of the linguistic quantifier. In addition, this approach is a generalization of the OWA-based evaluation.



**Keywords:** flexible querying, linguistic quantifiers, gradual numbers, OWA.


## 1 Introduction

Flexible querying of relational databases aims at expressing preferences into queries instead of boolean requirements as it is the case for regular (or crisp) querying. As a consequence, a flexible query returns a set of discriminated answers to the user (from the best answers to the less preferred). Many approaches to define flexible queries have been proposed and it has been shown that the fuzzy set based approach is the more general [1] and the more powerful. An extension of the SQL language (namely SQLf) has been proposed [2] to defined more sophisticated flexible queries calling on fuzzy sets.

In this context, predicates are defined by fuzzy sets and are called fuzzy predicates. They can be combined using various operators such as generalized conjunctions/disjunctions (expressed by norms/t-norms) or using more sophisticated operators such as averages. In addition, linguistic quantifiers [3] (which are quantifiers defined by linguistic expressions like "*most of*" or "*around 3*") allow to define a particular type of conditions called quantified statements. Such a condition is made of 3 elements : a linguistic quantifier (denoted by *Q*), a crisp set (denoted by X) and a fuzzy predicate (denoted by *A*) and means "*Q* elements belonging to X are *A*". A quantified statement is denoted "*Q* X are *A*" for short, as the quantified statement "*most of* employees are *well-paid*" (where *Q* is "*most of*", X a set of employees and *A* the condition "*to be well-paid*"). A second type of quantified statements can be distinguished where the quantifier applies to a fuzzy set as in "*most of young* employees are *well-paid*". This second type of statement is written "*Q B* X are *A*". Quantified statements can be used in flexible queries like in the query : "retrieve the firms where *most of* employees are *well-paid*". Each firm is associated to a degree in [0,1] expressing its satisfaction with respect to the quantified statement "*most of* employees are *well-paid*". The higher this degree, the better answer is the firm.

To evaluate a quantified statement is to determine the extent to which it is true. Many approaches have been proposed to evaluate statements of type "*Q* X are *A*". The most popular approaches (since meaningful) are the OWA operator [4] and the Sugeno fuzzy integral [5]. Both deliver a degree of truth and are limited to statements involving increasing quantifiers (as "*most of*", "*at least 3*").

This paper mainly considers the evaluation of quantified statements of type "*Q* X are *A*" in flexible querying where there is no particular reason to be limited to increasing quantifiers. As a consequence, its objective is to propose a framework to evaluate quantified statements of type "*Q* X are *A*" where *Q* can be of any type (with respect to monotonicity). This extension is based on the handling of fuzzy integers ($\mathbb{N}_f$, $\mathbb{Z}_f$) [6][7] and fuzzy rational numbers ($\mathbb{Q}_f$) as defined in [8]. These specific numbers express well-known but gradual numbers and differ from usual fuzzy numbers which define imprecise (ill-known) numbers. The proposed approach is also a generalization of the OWA-based interpretation of quantified statements.

Section 2 introduces the definition of linguistic quantifiers and the use of the OWA operator to evaluate quantified statements involving increasing quantifiers. A representation of linguistic quantifiers in terms of a tolerance with respect to a difference of fuzzy integers is given in section 3. Section 4 calls on this representation to propose an evaluation for "*Q* X are *A*" statements where *Q* is of any type. It is shown that when *Q* is increasing this evaluation reverts to the use of the OWA operator.

## 2 Linguistic quantifiers and the OWA operator

Two kinds of linguistic quantifiers can be distinguished: absolute quantifiers (which refer to an absolute number such as *about 3*, *at least 2*, ...) and relative quantifiers (which refer to a proportion such as *about the half*, *at least a quarter*, ...).

A linguistic quantifier can also be increasing (resp. decreasing), which means that an increase in the satisfaction to condition *A* cannot decrease (resp. increase) the truth value of the statement "*Q* X are *A*". *At least 3*, *almost all* (resp. *at most 2*, *at most the half*) are examples of increasing (resp. decreasing) quantifiers. A quantifier is monotonic when it is either increasing, or decreasing. It is also possible to point out unimodal quantifiers which refer to a quantity such as *about the half*, *about 4*.

The representation of an absolute quantifier is a fuzzy subset of the real line. A relative quantifier is defined by a fuzzy subset of the unit interval [0,1]. In both cases, the membership degree $\mu_Q(j)$ represents the truth value of the statement "*Q* X are *A*" when j elements in X completely satisfy *A*, whereas *A* is fully unsatisfied by the others (j being a number or a proportion). The representation of an increasing linguistic quantifier satisfies:

Def. (1) : i) $\mu_Q(0) = 0$, ii) $\exists$ k such as $\mu_Q(k) = 1$,
iii) $\forall$ a, b if $a > b$ then $\mu_Q(a) \geq \mu_Q(b)$

A decreasing linguistic quantifier is defined by:

Def. (2) : i) $\mu_Q(0) = 1$, ii) $\exists$ k such as $\mu_Q(k) = 0$,
iii) $\forall$ a, b if $a > b$ then $\mu_Q(a) \leq \mu_Q(b)$

A unimodal quantifier is a fuzzy subset *Q* such that:

Def. (3) : i) $\mu_Q(0) = 0$, ii) $\exists!$ k such as $\mu_Q(k) = 1$,
iii) $\forall$ $a < b < k$ then $\mu_Q(a) \leq \mu_Q(b)$
and $\forall$ $a > b \geq k$ then $\mu_Q(a) \leq \mu_Q(b)$.

***Example 1.*** Figures 1 and 2 are respectively representing an absolute increasing quantifier (*at least 3*, k = 3) and a relative unimodal quantifier (*about half*, k = 0.5) •

The interpretation of "*Q* X are *A*" (*Q* being increasing), by an ordered weighted average (OWA operator) is given by:

$$\text{OWA} = \sum_{i=1}^{n} (w_i * \mu_A(x_i)),$$

where the degrees of membership in *A* are decreasingly ranked $\mu_A(x_1) \geq \mu_A(x_2) \geq \ldots \geq \mu_A(x_n)$ and $w_i = \mu_Q(i) - \mu_Q(i-1)$ in case of an absolute quantifier and $\mu_Q(i/n) - \mu_Q((i-1)/n)$ in case of a relative one (n being the cardinality of set X). Each weight $w_i$ represents the increase in satisfaction when comparing a situation where (i-1) elements are entirely *A* with a situation where i elements are entirely *A*.

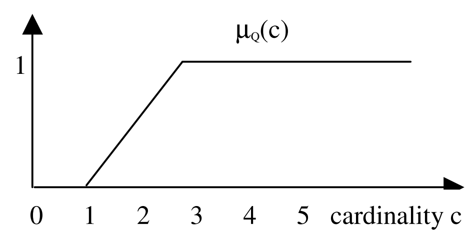


Figure 1: The quantifier *at least 3*

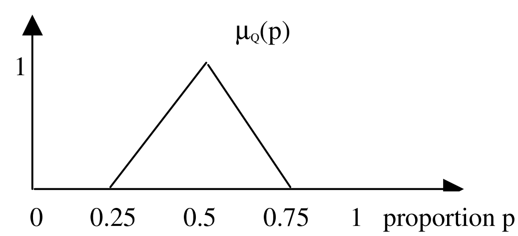


Figure 2: The quantifier *about half*

***Example 2.*** Let *Q* be the increasing absolute *at least 3* defined in figure 1 and $X = \{x_1, x_2, x_3, x_4\}$ with $\mu_A(x_1) = \mu_A(x_2) = 1$, $\mu_A(x_3) = 0.8$, $\mu_A(x_4) = 0.6$.

From figure 1, we get the weights:

$w_1 = \mu_Q(1) - \mu_Q(0) = 0$, $w_2 = \mu_Q(2) - \mu_Q(1) = 0.5$, $w_3 = \mu_Q(3) - \mu_Q(2) = 0.5$, $w_4 = \mu_Q(4) - \mu_Q(3) = 0$.

The interpretation of the statement "*almost all* X are *A*" by an OWA operator is:

$$(w_1 * \mu_A(x_1)) + (w_2 * \mu_A(x_2)) + (w_3 * \mu_A(x_3)) +$$

$(w_4 * \mu_A(x_4)) = (0.5 * 1) + (0.5 * 0.8) = 0.9.$

This result is close to 1, which means that "*at least 3* X are *A*" is rather true. This result fits the intuition since the three highest degrees of satisfaction with respect to A are 1, 1 and 0.8•

Another expression for the OWA operator is:

$$\text{OWA} = \sum_{i=1}^{n} \mu_Q(i) * (\mu_A(x_i) - \mu_A(x_{i+1})), \quad (1)$$

where the degrees of membership in *A* are decreasingly ranked $\mu_A(x_1) \geq \mu_A(x_2) \geq \ldots \geq \mu_A(x_n)$ and $\mu_A(x_{n+1}) = 0$. This equivalent expression is considered in subsection 3.2.3.

## 3 An interpretation based on fuzzy integers

A linguistic quantifier is defined by a fuzzy set expressing a constraint on crisp cardinalities (for absolute quantifiers) or crisp proportions (for relative quantifiers). As an example, the quantifier described in figure 1 expresses that a cardinality of 4 fully satisfies the constraint "*at least 3*" while a cardinality of 2 satisfies the same constraint at degree 0.5. When evaluating a "*Q* X are *A*" statement, the key point is to determine the extent to which a fuzzy quantity (either the fuzzy cardinality of the fuzzy set made of elements from X which are *A*, or the fuzzy proportion of *A* elements in X) satisfies the constraint described by *Q*. The approach suggested here uses another representation of linguistic quantifiers (subsection 3.1) and proposes to achieve computations (subsection 3.2) thanks to arithmetic operations on fuzzy integers ($\mathbb{N}_f$), fuzzy relative integers ($\mathbb{Z}_f$) [6] and fuzzy rational numbers ($\mathbb{Q}_f$) [8]. This new interpretation of quantified statements deals with any kinds of linguistic quantifiers (unimodal, increasing or decreasing).

### 3.1 A new representation for linguistic quantifiers

The fuzzy set which describes a linguistic quantifier can be decomposed into two parts, a value (denoted k in definitions Def. (1), Def. (2) and Def. (3)) which can be either an integer or a proportion and a tolerance with respect to this precise value. To obtain the tolerance it is necessary to shift the curve describing the quantifier. The tolerance function T is obtained using the translation $T(x\text{-}k) = \mu_Q(x)$. As a consequence, in the following, we denote a quantifier by a couple (k, T) made of the two items. As an example, the quantifier given by figure 1 is given by the following couple (3, T) where the tolerance is given by figure 3.

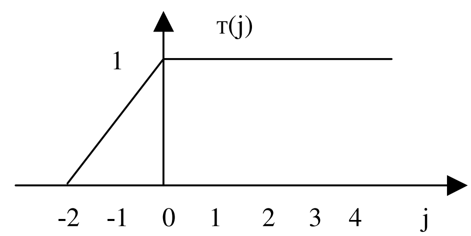


Figure 3: The tolerance function associated to the quantifier *at least 3*

In this last example, the tolerance function is defined on the set of integers (since the quantifier is absolute) but in case of a relative quantifier, the tolerance function is defined on [0, 1] (on proportions).

### 3.2 Interpreting quantified statements using fuzzy arithmetic

In this subsection, we consider the evaluation of quantified statements of type "*Q* X are *A*" where *Q* is given by the new representation introduced in the previous subsection. No particular assumption is made on the monotonicity of the quantifier. We consider in 3.2.1 the case of an absolute quantifier while the case of a relative integer is introduced in 3.2.2. It is then shown in 3.2.3 that when *Q* is increasing, the process introduced here reverts to the interpretation given by the OWA operator.

3.2.1 Absolute quantifier

First, the case where *A* is a crisp predicate is presented. It is then extended to the general case where *A* is fuzzy.

When *A* is a crisp predicate, the cardinality of the fuzzy set made of elements from X which are *A* is a precise number (denoted c in the following). The quantified statement "*Q* X are *A*" is then evaluated by $\mu_Q(c)$. When referring to the new representation of quantifiers, it immediately gives T(c-k) (since $\forall x$, $T(x\text{-}k) = \mu_Q(x)$). Same computations are retained in case of a fuzzy predicate for *A*. First, (c-k) is computed (c being an FGCount) and T(c-k) is represented by a fuzzy set. This fuzzy set is defuzzified to provide a scalar value for T(c-k).

When *A* is a fuzzy set, the cardinality is described by a fuzzy number given by:

$$\forall\, e \in \mathbb{N},\ \mu_c(e) = \sup \{\alpha \,/\, |A_\alpha| \geq e\}.$$

This definition is that of the FGCount(*A*) proposed by Zadeh [3]. The degree $\mu_c(e)$ expresses the extent to which there is at least e elements in fuzzy set *A*. Fuzzy number c is a normalized convex conjunctive fuzzy set which is non decreasing.

***Example 3.*** If we consider the fuzzy set *A* given in example 2, we get:

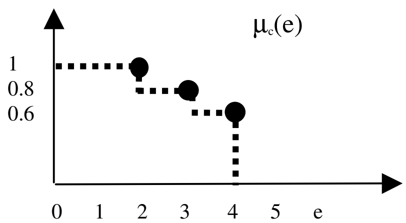


Figure 4: A fuzzy integer

•

The computation of T(c-k) is made thanks to the operations defined for fuzzy integers developed in [6]. The key point is that each fuzzy natural integer ($\mathbb{N}_f$) is a FGCount (similarly to the crisp integers defined as crisp cardinalities) while fuzzy relative integers ($\mathbb{Z}_f$) have been defined as couples of fuzzy natural integers.

As we first need to compute (c-k), where c belongs to $\mathbb{N}_f$, it is necessary to represent the crisp integer k as a fuzzy integer (to compute the difference between two fuzzy numbers). More precisely, k is defined as the FGCount of any crisp set having exactly k elements:

$$\forall\, e \in [0,k],\ \mu_k(e) = 1,$$
$$\forall\, e > k,\ \mu_k(e) = 0.$$

The difference (c-k) belongs to $\mathbb{Z}_f$ and is represented by the couple (c, k) (see [] for details). A canonical representation of (c-k) can be obtained by considering each level $\alpha$:

$$\forall\, \alpha \in [0, 1],\ \Delta(\alpha) = c(\alpha) - k(\alpha),$$

where $c(\alpha)$ (resp. $k(\alpha)$) is the cardinality of the $\alpha$ level cut of any fuzzy sets having c (resp. k) as fuzzy cardinality. It gives:

$$\forall\, \alpha \in [0, 1],\ \Delta(\alpha) = |A_\alpha| - k.$$

Moreover, applying a predicate T on a fuzzy relative integer such as $\Delta$ leads to apply T on the cardinalities of the different $\alpha$-cuts of the underlying fuzzy sets [9]. Consequently the evaluation of $T(\Delta)$ is performed $\alpha$-cut by $\alpha$-cut and the satisfaction of (c-k) to tolerance T is given by:

$$\begin{aligned} \forall\, \alpha \in [0, 1],\ \mu_{T(c\text{-}k)}(\alpha) &= T(\Delta(\alpha)), \\ &= T(|A_\alpha| - k), \\ &= \mu_Q(|A_\alpha|). \end{aligned} \qquad (2)$$

The fuzzy set T(c-k) is a fuzzy truth value expressing the satisfaction of each $\alpha$-cut of *A* with respect to the linguistic quantifier .

***Example 4.*** If we consider the fuzzy set *A* given in example 2 and the statement "*about 3* X are *A*" where the linguistic quantifier *about 3* is given by figure 5.

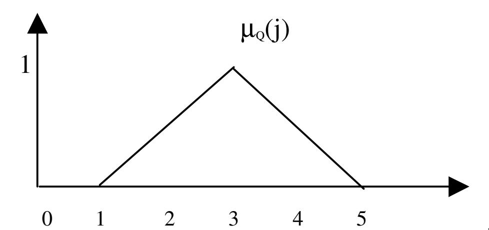


Figure 5: A representation for the quantifier *about 3*

We get the following fuzzy truth value for "*about 3* X are *A*" :

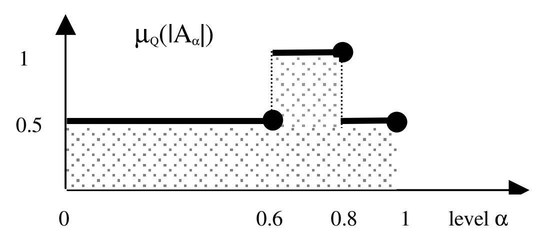


Figure 6: A fuzzy truth value

•

As $\alpha$-cuts are different interpretations of fuzzy set *A*, one may remark that the more $\alpha$-cuts satisfy the constraint defined by *Q*, the more fuzzy set *A* satisfies the constraint defined by *Q*. It means the following degree of satisfaction $\delta$ can be computed for "*Q* X are *A*" :

$$\delta = \int \mu_{T(c\text{-}k)}(\alpha)\, d\alpha$$

Value $\delta$ is the area delimited by function $\mu_{T(c-k)}$. Since this function is a stepwise function, we get :

$$\delta = (\alpha_1 - 0) * \mu_{T(c-k)}(\alpha_1) + (\alpha_2 - \alpha_1) * \mu_{T(c-k)}(\alpha_2) + \ldots + (1 - \alpha_n) * \mu_{T(c-k)}(1), \quad (3)$$

where the discontinuity points are $(\alpha_1, \mu_{T(c-k)}(\alpha_1))$, $(\alpha_2, \mu_{T(c-k)}(\alpha_2))$, ..., $(\alpha_n, \mu_{T(c-k)}(\alpha_n))$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. When $\delta = 1$, it means that any α-cut of fuzzy set *A* fully satisfies the constraint defined by the quantifier. The higher δ, the more α-cuts highly satisfy the constraint "*Q* elements are *A*". In addition, one may remark that the $\alpha_i$'s are nothing but the different degrees of membership in *A*.

***Example 5.*** When referring to the previous example, from figure 6 we compute:

$$\delta = (0.6 - 0) * 0.5 + (0.8 - 0.6) * 1 + (1 - 0.8) * 0.5, = 0.6.$$

The result is coherent with the intuition since it seems that the cardinality of fuzzy set *A* is between 2 and 3 with $\mu_Q(2) = 0.5$ and $\mu_Q(3) = 1$ •

However, one may point out that different definitions can be envisaged for δ since it is the result of a defuzzification (see [9]).

3.2.2 Relative quantifier

The interpretation of "*Q* X are *A*" in case of a relative quantifier can be adapted from that an absolute quantifier by considering the cardinality (denoted n hereafter) of crisp set X and a fuzzy proportion instead of a fuzzy cardinality.

When *A* is a crisp predicate, the quantified statement "*Q* X are *A*" is evaluated by $\mu_Q(c/n)$ where c is the cardinality of the crisp set made of element from X which are *A*. When referring to the new representation of quantifiers, it immediately gives T((c/n)-k) (since $\forall x \in [0, 1], T(x-k) = \mu_Q(x)$).

When *A* is a fuzzy set, (c/n)-k is a fuzzy rational number (from $\mathbb{Q}_f$) and T((c/n)-k) is represented by a fuzzy set defined by:

$$\forall \alpha \in [0, 1], \mu_{T((c/n)-k)}(\alpha) = \mu_Q(|A_\alpha|/n). \quad (4)$$

Here again, the fuzzy value T((c/n)-k) expresses the satisfaction of each α-cut with respect to the linguistic quantifier and one may remark that the more α-cuts satisfy the constraint defined by *Q*, the more fuzzy set *A* satisfies the constraint defined by *Q*. Similarly to the case of an absolute quantifier, an overall degree of satisfaction δ can be computed for "*Q* X are *A*":

$$\delta = \int \mu_{T((c/n)-k)}(\alpha)\, d\alpha$$

Value δ is the area delimited by function $\mu_{T((c/n)-k)}$. By denoting $\alpha_i$'s the different degrees of membership in *A* with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$., we get:

$$\delta = (\alpha_1 - 0) * \mu_{T((c/n)-k)}(\alpha_1) + (\alpha_2 - \alpha_1) * \mu_{T((c/n)-k)}(\alpha_2) + \ldots + (1 - \alpha_n) * \mu_{T((c/n)-k)}(1) \quad (5)$$

Thus, the higher δ, the more α-cuts highly satisfy the constraint "*Q* elements are *A*". In addition, one may remark that the $\alpha_i$'s are nothing but the different degrees of membership in *A*.

***Example 6.*** We consider the fuzzy set *A* given in example 2 and the statement "*about half* X are *A*" where the linguistic quantifier *about half* is given by figure 2. From definition (3) we get the following fuzzy truth value for "about half X are *A*":

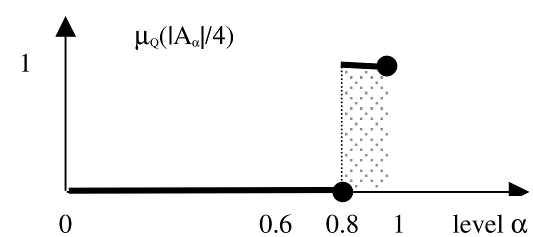


Figure 7: A fuzzy truth value

We obtain:

$$\delta = (1 - 0.8) * 1 = 0.2.$$

The result is coherent with the intuition since it seems that the proportion of *A* elements in X seems to be a little bit less than 3/4 •

3.2.3 Relation with the OWA operator

This subsection shows that, when the quantifier is increasing, the approach advocated here to interpret "*Q* X are *A*" statements leads to an OWA operator.

We demonstrate that expression (3) (in case of an increasing absolute quantifier) and expression (5) (in case of an increasing relative quantifier) are OWA operators.

In expression (3) and (5) the $\alpha_i$'s are nothing but the different degrees of membership in *A*. It means expression (3) (resp. (5)) can be rewritten:

$\delta = (\mu_A(x_n) - 0) * \mu_{T(c-k)}(\mu_A(x_n)) + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_{T(c-k)}(\mu_A(x_{n-1})) + ... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_{T(c-k)}(\mu_A(x_1)) + (1 - \mu_A(x_1)) * \mu_{T(c-k)}(1)$,

(resp. $\delta = (\mu_A(x_n) - 0) * \mu_{T((c/n)-k)}(\mu_A(x_n)) + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_{T((c/n)-k)}(\mu_A(x_{n-1})) + ... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_{T((c/n)-k)}(\mu_A(x_1)) + (1 - \mu_A(x_1)) * \mu_{T((c/n)-k)}(1)$),

where $\mu_A(x_1) \geq \mu_A(x_2) \geq ... \geq \mu_A(x_n)$. As $\mu_{T(c-k)}(\alpha) = \mu_Q(|A_\alpha|)$ (resp. $\mu_{T((c/n)-k)}(\alpha) = \mu_Q(|A_\alpha|/n)$), we can write :

$\delta = (\mu_A(x_n) - 0) * \mu_Q(n), + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_Q(n-1) + ... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_Q(1) + (1 - \mu_A(x_1)) * \mu_Q(|A_1|)$,

(resp. $\delta = (\mu_A(x_n) - 0) * \mu_Q(1), + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_Q(n-1/n) + .... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_Q(1/n) + (1 - \mu_A(x_1)) * \mu_Q(|A_1|/n)$).

However $(1 - \mu_A(x_1)) * \mu_Q(|A_1|)$ (resp. $(1 - \mu_A(x_1)) * \mu_Q(|A_1|/n)$) is 0:

- when A is normalized $\mu_A(x_1) = 1$,
- when *A* is not normalized $\mu_Q(|A_1|) = \mu_Q(|A_1|/n) = \mu_Q(0) = 0$, *Q* being increasing.

and so:

$\delta = (\mu_A(x_n) - 0) * \mu_Q(n), + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_Q(n-1) + ... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_Q(1)$,

(resp. $\delta = (\mu_A(x_n) - 0) * \mu_Q(n/n), + (\mu_A(x_{n-1}) - \mu_A(x_n)) * \mu_Q(n-1/n) + .... + (\mu_A(x_1) - \mu_A(x_2)) * \mu_Q(1/n)$).

which is nothing but the operator OWA (expression (1)).

## 4. Conclusion

This paper takes place at the crossword of flexible querying of relational databases using fuzzy sets and fuzzy arithmetic introduced in [6][9][10]. It shows that fuzzy arithmetic allows to evaluate quantified statements involving any kind of quantifiers (with respect to monotonicity). The proposed approach is a generalization of the OWA-based interpretation (this one is limited to monotonic quantifiers). It focuses on statements of type "*Q* X are *A*", but fuzzy arithmetic also provides a sound basis to evaluate statements of type "*Q B* X are *A*". As an example, the statement "*most of young* employees are *well-paid*" could be evaluated by confronting the fuzzy ratio "cardinality of *young* and *well-paid* employees / cardinality of *young* employees" with the fuzzy number "*around 1*".

In the near future, we aim at comparing a fuzzy arithmetic based approach to evaluate quantified statements with the Sugeno Fuzzy Integral based interpretation.

## References


[1] Bosc, P. & Pivert, O. "Some Approaches for Relational Databases Flexible Querying", *Journal of Intelligent Information Systems*, 1, 323-354, 1992.

[2] Bosc, P. & Liétard, L. & Pivert, O., "Quantified statements and database fuzzy querying ", *Fuzziness in Database Management Systems*, 275-308, 1995.

[3] Zadeh, L.A, "A computational approach to fuzzy quantifiers in natural languages", Computer Mathematics with Applications, 9, 149-183, 1983.

[4] Yager, R.R., "On ordered weighted averaging aggregation operators in multicriteria decisionmaking". *Transac. on Syst., Man, and Cybernetics*, 18, 183-190, 1988.

[5] Bosc, P., Liétard, L., Pivert, O., "Sugeno fuzzy integral as a basis for the interpretation of flexible queries involving monotonic". *Information processing and management*, 39, 287-306, 2003

[6] Rocacher D., Bosc P., "About Zf, the set of fuzzy relative integers, and the definition of fuzzy bags on Zf," Lecture Notes in Computer Science, LNCS 2715, Springer-Verlag, 95-102, 2003.

[7] Rocacher D., Bosc P., "Entiers relatifs flous et multi-ensembles flous", *Rencontres francophones sur la logique floues et ses applications*, (LFA'03), 253-260, 2003.

[8] Rocacher D., Bosc P., "Sur la définition des nombres rationnels flous", *Rencontres francophones sur la logique floues et ses applications*, (LFA'03), 261-268, 2003

[9] Rocacher D., Bosc P., "Relations d'ordre floues sur des quantités floues et expression de requêtes flexibles," *Actes des Rencontres Francophones sur la Logique Floue et ses Applications* (LFA'04), 2004.

[10] Rocacher D., Bosc P., "The set of fuzzy rational numbers and flexible querying", Fuzzy Sets and Systems, to appear.