# Evaluation of Quantified Statements using Gradual Numbers

Ludovic Liétard*

IRISA/IUT
rue Edouard Branly
BP 30219
Lannion cedex France
ludovic.lietard@univ-rennes1.fr

Daniel Rocacher*

IRISA/ENSSAT
rue de Kerampont
BP 447
Lannion cedex France
daniel.rocacher@enssat.fr

Dr. Ludovic Liétard is currently assistant professor at the University of Rennes 1 (IUT Lannion) in France. His research mainly concerns flexible querying of relational databases using fuzzy set theory and various applications of fuzzy set theory in databases.

Dr. Daniel Rocacher is currently assistant professor at the University of Rennes 1 (ENSSAT Lannion) in France. He has proposed new directions to define gradual numbers in the framework of fuzzy set theory. His current research concerns their applications in databases.

Abstract.

This paper is devoted to the evaluation of quantified statements which can be found in many applications as decision-making, expert systems or flexible querying of relational databases using fuzzy set theory. Its contribution is to introduce the main techniques to evaluate such statements and to propose a new theoretical background for the evaluation of quantified statements of type "*Q X* are *A*" and "*Q B X* are *A*". In this context, quantified statements are interpreted using an arithmetic on gradual numbers from $\mathbb{N}_f$, $\mathbb{Z}_f$ and $\mathbb{Q}_f$. It is shown that the context of fuzzy numbers provides a framework to unify previous approaches and can be the base for the definition of new approaches.

## INTRODUCTION

Linguistic quantifiers are quantifiers defined by linguistic expressions like "*around 5*" or "*most of*" and many types of linguistic quantifiers can be found in the litterature (Diaz-Hermida et al., 2003) or (Losada et al., 2006) or (Glockner, 1997, 2004a, 2004b) (as semi-fuzzy quantifiers which allow to model expressions like "there are *twice as many* men *as* women"). We limit this presentation to the original linguistic quantifiers defined by Zadeh (1983) and the two types of quantified statements he proposes.

Such linguistic quantifiers allow an intermediate attitude between the conjunction (expressed by the universal quantifier $\forall$) and the disjunction (expressed by the existential quantifier $\exists$). Two types of quantified statements can be distinguished. A statement of the first type is denoted "*Q X* are *A*" where *Q* is a linguistic quantifier, *X* is a crisp set and *A* is a fuzzy predicate. Such a statement means that "*Q* elements belonging to *X* satisfy *A*". An example is provided by "*most of* employees are *well-paid*" where *Q* is *most of*, *X* is a set of employees whereas *A* is the condition *to be well-paid*. In this first type of quantified statements, the referential (denoted by *X*) for the linguistic quantifier is a crisp set (a set of employees in the example). A second type of quantified statements can be defined where the linguistic quantifier applies to a fuzzy referential. This is the case of the statement "*most of young* employees are *well-paid*" since *most of* applies to the fuzzy referential made of *young* employees. This statement means that *most of* elements from this fuzzy referential (*most of young* employees) can be considered *well-paid*. Such a quantified statement is written "*Q B X* are *A*" where *A* and *B* are two fuzzy predicates (when referring to the previous example, *Q* is *most of*, *X* is a set of employees, *B* is *to be young* while *A* is *to be well-paid*).

Linguistic quantifiers can be used in many fields and we briefly recall their use in multicriteria decision-making, expert systems, linguistic summaries of data and flexible querying of relational databases (some minor applications of linguistic quantifiers as in machine learning (Kacprzyck & Iwanski, 1992), or neural networks (Yager, 1992) are not dealt with).

Multicriteria decision-making consists mainly in finding optimal solutions to a problem defined by objectives and constraints. A solution must fullfill *all* objectives and must satisfy *all* constraints. The use of linguistic quantifiers in decision-making (Yager, 1983a) or (Kacprzyck, 1991) or (Malczewski & Rinner, 2005) or (Fan, Z.P. & Chen, X., 2005) aims at retrieving solutions fullfilling Q objectives with respect to Q' constraints, where Q and Q' are either a linguistic quantifier or the universal quantifier. A typical formulation is then "find the solution where *almost all* objectives are achieved and where *all* constraints are satisfied.

The use of linguistic quantifiers in expert systems concerns mainly the expression and handling of logical propositions. An example is provided by logical statements accepting exceptions. A typical statement accepting exceptions is the proposition "*all* Sweden are tall" which can be turned into "*almost all* Sweden are tall" involving the linguistic quantifier "*almost all*". Many inferences involving quantified statements are possible (Mizumoto et al., 1979) or (Dubois & Prade, 1988a) or (Dubois et al., 1993) or (Sanchez, 1988) or (Laurent et al., 2003) or (Loureiro Ralha & Ghedini Ralha, 2004). It is possible to consider the following one, set in the probabilistic framework : if I know that "Karl is Sweden" and that "*almost all* Sweden are tall", it is then possible to infer that the event "Karl is tall" is probable. The challenge is then to compute the degree of probability (which may be imprecise) attached to the event "Karl is tall".

Data sumarization (Sicilia et al., 2002) or (Kacprzyck, 2006) is another field where linguistic quantifiers can be helpful. R.R. Yager (1982) defines summaries expressed by expressions involving linguistic quantifiers (the summary of a database could be "*almost the half young* employees are *well-paid*"). SummarySQL language (Rasmussen & Yager, 1997) has been proposed to define and evaluate linguistic summaries of data defined by quantified statements. As an example, it is possible to use this language to determine the validity (represented by a degree) on a given database of the linguistic summary *"almost the half young* employees are *well-paid*".

Flexible querying of relational databases aims at expressing preferences into queries instead of boolean requirements as it is the case for regular (or crisp) querying. Consequently, a flexible query returns a set of discriminated answers to the user (from the best answers to the less preferred). Many approaches to define flexible queries have been proposed and it has been shown that the fuzzy set based approach is the more general (Bosc & Pivert, 1992). Extensions of the SQL language, namely SQLf (Bosc & Pivert, 1995), and FSQL (Galindo et al., 1998) and (Galindo et al., 2006) and (Galindo, 2005, 2007) have been proposed to define sophisticated flexible queries calling on fuzzy sets (in this book, reader can find a chapter by Urrutia, Tineo and Gonzalez including a comparison between FSQL and SQLf). In this context, predicates are defined by fuzzy sets and are called fuzzy predicates and they can be combined using various operators such as generalized conjunctions and generalized disjunctions (respectively expressed by norms and t-norms) or using more sophisticated operators such as averages. Fuzzy predicate can also be defined by a quantified statements, as in the query: "retrieve the firms where *most of* employees are *well-paid*". After query evaluation, each firm is associated to a degree in *[0,1]* expressing its satisfaction with respect to the quantified statement of the first type: "*most of* employees are *well-paid*". The higher this degree, the better answer is the firm.

To evaluate a quantified statement is to determine the extent to which it is true. This paper proposes a new theoretical framework to evaluate quantified statements of type "*Q X* are *A*" and "*Q B X* are *A*". Propositions are based on the handling of gradual integers (from $\mathbb{N}_f$ and $\mathbb{Z}_f$) (Rocacher & Bosc, 2003a, 2003b) and gradual rational numbers (from $\mathbb{Q}_f$) as defined in (Rocacher & Bosc, 2003c, 2005). These specific numbers express well-known but gradual numbers and differ from usual fuzzy numbers which define imprecise (ill-known) numbers.

Section "Linguistic quantifiers and quantified statements" introduces the definition of quantified statements while section "Previous proposals for the interpretation of quantified" is a brief overview of the proposition made for the evaluation of quantified statements. Gradual numbers are introduced in section "GRADUAL NUMBERS AND GRADUAL TRUTH VALUE" and section "INTERPRETATION OF QUANTIFIED STATEMENTS USING GRADUAL NUMBERS" proposes to evaluate quantified statements using gradual numbers. In the following, we denotes *A(X)* the fuzzy set made of elements from a crisp set *X* which satisfy a fuzzy predicate *A* (*A(X)* being defined by $X \cap A$).

## LINGUISTIC QUANTIFIERS AND QUANTIFIED STATEMENTS

First order logic involves two quantifiers, the universal quantifier ($\forall$) and the existential one ($\exists$), which are too limited to model all natural language quantified sentences. For this reason, fuzzy quantifiers (Zadeh, 1983) have been introduced to represent linguistic expressions (*many of*, *at least 3*…) and to refer to gradual quantities.

It is possible to distinguish between absolute quantifiers (which refer to an absolute number such as *about 3*, *at least 2*, ...) and relative quantifiers (which refer to a proportion such as *about the half*, *at least a quarter*, ...). An absolute (resp. relative) quantifier *Q* in the statement "*Q X* are *A*" means that the number (resp. proportion) of elements satisfying condition *A* is compatible with *Q*.

A linguistic quantifier can be increasing (resp. decreasing) (Yager, 1988) which means that an increase in the satisfaction to condition *A* cannot decrease (resp. increase) the truth value of the statement "*Q X* are *A*". *At least 3*, *almost all* (resp. *at most 2*, *at most the half*) are examples of increasing (resp. decreasing) quantifiers. A quantifier is monotonic when it is either increasing, or decreasing and it is also possible to point out unimodal quantifiers which refer to a quantity such as *about the half*, *about 4*, etc.

The representation of an absolute quantifier is a fuzzy subset of the real line while a relative quantifier is defined by a fuzzy subset of the unit interval $[0,1]$. In both cases, the membership degree $\mu_Q(j)$ represents the truth value of the statement "*Q X* are *A*" when $j$ elements in *X* completely satisfy *A*, whereas *A* is fully unsatisfied by the others ($j$ being a number or a proportion). In other words, the definition of a linguistic quantifier provides the evaluation for "*Q X* are *A*" in case of a Boolean predicate.

Consequently, the representation of an increasing (resp. decreasing) linguistic quantifier is an increasing (resp. decreasing) function $\mu_Q$ such that $\mu_Q(0) = 0$ (resp. $\mu_Q(0) = 1$) and $\exists k$ such as $\mu_Q(k) = 1$ (resp. $\exists k$ such as $\mu_Q(k) = 0$).

*Example.* Figure 1 describes the increasing relative linguistic quantifier *almost all*.

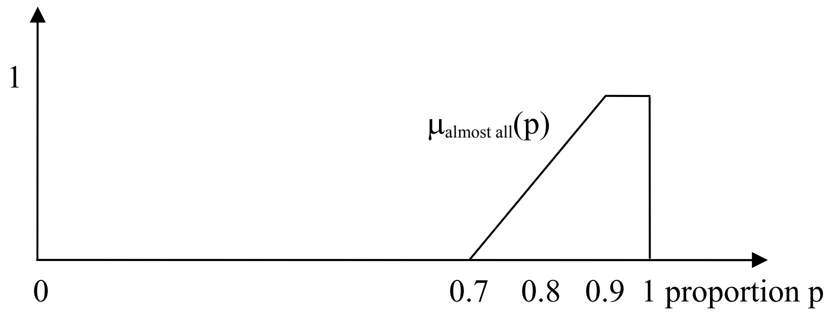


**Figure 1**. A representation for the quantifier *almost all*•

It is worth mentioning that, in case of an absolute quantifier, a quantified statement of type "*Q B X* are *A*" reverts to the quantified statement of the other type : "*Q X* are (*A and B)*". As an example, "*at least 3 young* employees are *well-paid*" is equivalent to "*at least 3* employees are (*young* and *well-paid*)". As a consequence, when dealing with quantified statements of type "*Q B X* are *A*", this paper only deals with relative quantifiers.

## PREVIOUS PROPOSALS FOR THE INTERPRETATION OF QUANTIFIED STATEMENTS

In this section, the main propositions suggested to determine the truth value of quantified statements are briefly overviewed. An in-depth study of quantified statements interpretations can be found in (Liu & Kerre, 1998a, 1998b) or (Delgado et al., 2000) or (Barro et al., 2003) or (Diaz-Hermida et al., 2004). Subsection “Quantified statements of type

"*Q X* are *A*"" is devoted to the evaluation of quantified statements of type "*Q X* are *A*" whereas subsection "Quantified statements of type "*Q B X* are *A*"" is devoted to the evaluation of quantified statements of type "*Q B X* are *A*". A short conclusion about these proposals is provided in subsection "About the proposed approaches to evaluate quantified statements".

Quantified statements of type "*Q X* are *A*"

Relative quantifiers are assumed hereafter and the adaptation to absolute quantifiers requires only the change of the quantity $\mu_Q(i/n)$ into $\mu_Q(i)$, $n$ being the cardinality of set $X$ involved in the quantified statement.

In the particular case of a Boolean predicate *A,* the evaluation of "*Q X* are *A*" is given by $\mu_Q(c)$ where $c$ is the number of elements satisfying *A*. Some approaches (interpretations based on a precise and an imprecise cardinality) extend this definition to a fuzzy predicate *A* assuming that the cardinality of a fuzzy set can be computed. Other approaches (using an OWA operator or a Sugeno fuzzy integral) are based on a relaxation principle which implies the neglection of some elements. As an example, the interpretation of "*almost all* employees are *young*" means that some oldest employees can be (more or less) neglected before assessing the extent to which the remaining employees are *young*.

*Interpretation based on a precise cardinality*

Zadeh (1983) suggests to compute the precise cardinality of fuzzy set A (called sigma-count and denoted $\sum Count(A)$). The sigma-count is defined as the sum of membership degrees and the degree of truth of "*Q X* are *A*" is then $\mu_Q(\sum Count(A)/n)$, n being the cardinality of set *X*.

The definition of $\Sigma Count(A)$ implies that a large number of small $\mu_A(x)$ values has the same effect on the result than a small number of large $\mu_A(x)$ values. As a consequence, many drawbacks can be found, as the one shown by the next example.

*Example.* Set $X = \{x_1, x_2, ..., x_{10}\}$ is such that $\forall i,\ \mu_A(x_i) = 0.1$. In this case, the result for "$\exists X$ are *A*" is expected to be *0.1* (or at least extremely low). The existential quantifier $\exists$ is defined by $\mu_\exists(0) = 0$ and $\forall i > 0,\ \mu_\exists(i) = 1$ and the absolute quantified statement is evaluated by $\mu_\exists(\sum Count(A))$. Computations give $\sum Count(A) = 1$ which implies that expression "$\exists$ *X* are *A*" is entirely true $(\mu_\exists(1) = 1)$. This result is very far from the expected one. ♦

*Interpretation based on an imprecise cardinality*

The method proposed in (Prade, 1990) involves two steps. The first one computes the imprecise cardinality $\pi c$ of the set made of elements from *X* which satisfy *A* (it is a fuzzy number represented by a possibility distribution of integers). Then, quantifier *Q* is considered a vague predicate serving as a basis for a matching with $\pi c$. The result is a couple of degrees, the possibility and the necessity of the fuzzy event "$\pi c$ is compatible with $Q$".

The imprecise cardinality of the set F of elements from $X$ which satisfy $A$, is given by the following possibility distribution (Dubois & Prade, 1985) and (Prade 1990) :

let k be the number of values of F whose degree is $1$: $\pi c(k) = 1$ ($k$ may equal $0$),
$\forall i < k,\ \pi c(i) = 0,$
$\forall j > k,\ \pi c(j)$ is the $j^{th}$ largest value $\mu_F(x)$.

In the particular case where F is a usual set, $\pi c$ describes a precise value ($\pi c(k) = 1$ and $\pi c(i) = 0\ \ \forall i \neq k$) which is the usual cardinality of this set.

The possibility $\Pi(Q\ ;\ \pi c)$ and the necessity $N(Q\ ;\ \pi c)$ of the fuzzy event "$\pi c$ is compatible with $Q$" are (Dubois et al., 1988b)

$\Pi(Q\ ;\ \pi c) = \max_{1 \leq i \leq n} \min(\mu_Q(i/n), \pi c(i))$ and
$N(Q\ ;\ \pi c) = \min_{1 \leq i \leq n} \max(\mu_Q(i/n), 1 - \pi c(i)).$

*Example.* Let $Q$ be the increasing relative quantifier *almost all* defined in figure 2 and $X = \{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}\}$ with $\mu_A(x_1) = \mu_A(x_2) = \ldots = \mu_A(x_7) = 1,\ \mu_A(x_8) = 0.9,$ $\mu_A(x_9) = 0.7,\ \mu_A(x_{10}) = 0.$

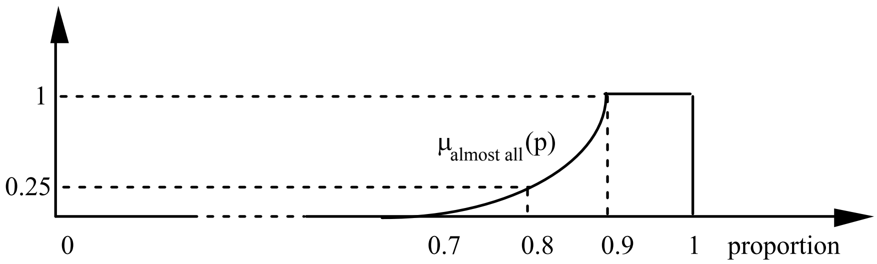


**Figure 2**. A representation for the quantifier *almost all*

We have:

$$\pi c(7) = 1,\ \pi c(8) = 0.9 \text{ and } \pi c(9) = 0.7,$$

with:

$$\mu_{almost\ all}(1/10) = \ldots = \mu_{almost\ all}(7/10) = 0,\ \mu_{almost\ all}(8/10) = 0.25,\ \mu_{almost\ all}(9/10) = \mu_{almost\ all}(1) = 1.$$

The interpretation of "*almost all X* are *A*" leads to:

$$\Pi(Q\ ;\ \pi c) = \max \min(\mu_{almost\ all}(7/10),\ \pi c(7)),\ \max(\mu_{almost\ all}(8/10),\ \pi c(8)),\ \max(\mu_{almost\ all}(9/10),\ \pi c(9))$$

$$= \max \min(0, 1),\ \min (0.25, 0.9),\ \min (1, 0.7) = 0.7,$$

$$N(Q\ ;\ \pi c) = \min \max(\mu_{almost\ all}(7/10),\ 1 - \pi c(7)),\ \max(\mu_{almost\ all}(8/10),\ 1 - \pi c(8)),\ \max(\mu_{almost\ all}(9/10),\ 1 - \pi c(9))$$

$$= \min \max (0 , 0),\ \max (0.25, 0.1),\ \max(1, 0.3) = 0 \ \blacklozenge$$

*Interpretation by the OWA operator*

We assume that $X = \{x_1, ..., x_n\}$ and $\mu_A(x_1) \geq \mu_A(x_2) \geq ... \geq \mu_A(x_n)$. The interpretation of "*Q X* are *A*" (*Q* being increasing), by an ordered weighted average (OWA operator) is given by (Yager, 1988):

$$\sum_{i=1}^{n}(w_i * \mu_A(x_i)),$$

where $w_i = \mu_Q(i/n) - \mu_Q(i-1/n)$. Each weight $w_i$ represents the increase of satisfaction when comparing a situation where (*i-1*) elements are entirely *A* with a situation where *i* elements are entirely *A* (and the others are not at all *A*). This operator conveys a semantics of relaxation since *the smaller* $w_i$, *the more neglected* $\mu_A(x_i)$.

An extension of the use of the OWA operator to decreasing quantifiers has been proposed by Yager (Yager, 1993) and Bosc and Liétard (Bosc & Liétard, 1993). The extension is based on the equivalence:

$$\text{"}Q\ X \text{ are } A\text{"} \Leftrightarrow \text{"}Q'\ X \text{ are } \overline{A}\text{"},$$

where *Q'* is the antonym of the decreasing quantifier *Q* (*Q*' is then an increasing quantifier given by $\forall p \in [0,1],\ \mu_{Q'}(p) = \mu_Q(1-p))$. It is then possible to use the initial proposition to interpret "$Q'X$ are $\overline{A}$".

In addition, when $Q$ is not monotonic, this approach leads to the GD method introduced in section "The probabilistic approach (GD method)".

*The probabilistic approach (GD method)*

This method (Delgado et al., 2000) is based on the following imprecise cardinality of the fuzzy set $A(X)$ :

$\forall k \in \{0,1,2,.. n\}, p(k) = b_k - b_{k+1},$

where n is the cardinality of set $X$ and $b_k$ is the $k^{th}$ largest value of belongingness of an element to the fuzzy set $A(X)$ (with $b_0 = 1$ and $b_{n+1} = 0$). A value $p(k)$ can be interpreted as the probability that set $A(X)$ contains $k$ elements. The evaluation of a "$Q$ $X$ are $A$" statement with an absolute quantifier is :

$$\sum_{k=0}^{n} p(k) \times \mu_Q(k)$$

When $Q$ is relative, the evaluation becomes :

$$\sum_{k=0}^{n} p(k) \times \mu_Q(k/n).$$

This interpretation is clearly the average value of the different values taken by the linguistic quantifier.

*The ZS method*

The ZS method proposed in (Delgado et al., 2000) considers the following fuzzy cardinality $\pi$ of the fuzzy set $A(X)$ of elements which satisfy predicate $A$ :

$\pi(k) = 0$ if it does not exist a level cut $\alpha$ such that $|A(X)_\alpha| = k$,

otherwise $\pi(k) = sup\{\alpha \text{ such that } |A(X)_\alpha| = k\}$ .

This fuzzy cardinality can be interpreted as a possibility. The interpretation $\delta$ of the quantified statement "*Q X* are *A*" is the compatibility of the fuzzy quantifier *Q* with that fuzzy cardinality:

$$\delta = max_{1 \leq k \leq n}\, min\, (\mu_Q(k),\ \pi(k)),$$

where *n* is the cardinality of set *X*.

This evaluation clearly provides the possibility of the event "the cardinality satisfies *Q*" (as in the approach briefly introduced in 3.1.2). In addition, it is a generalization (Delgado et al., 2000) of the Sugeno fuzzy integral approach since when *Q* is increasing, the ZS and the Sugeno integral methods lead to a same result.

*Interpretation based on a Sugeno fuzzy integral*

The interpretation of "*Q X* are *A*" (*Q* being increasing) by a Sugeno fuzzy integral (Bosc & Liétard, 1994a, 1994b) or (Ying 2006) is given by:

$$\delta = max_{1 \leq i \leq n} min (\mu_Q(i), \mu_A(x_i)),$$

where $\mu_A(x_1) \geq \mu_A(x_2) \geq ... \geq \mu_A(x_n)$. Due to the properties of the Sugeno fuzzy integral, $\delta$ states the existence of a subset *C* of *X* such that:

- each elements in *C* is *A* with some concrete degree,
- subset *C* is in agreement with the linguistic quantifier *Q*.

Since *Q* is increasing, the more these two aspects are met, the higher the truth value for "*Q X* are *A*". As an example, "*almost all* employees are *young*" is evaluated by the existence of a subset of *young* employees which gathers *almost all* the employees. More precisely, δ can also be defined by:

$$\delta = max_{C \in P(X)} min(p_1(C), p_2(C)),$$

where *P*(*X*) denotes the powerset of *X* and $p_1(C)$ is defined by $min_{x \in C} \mu_A(x)$, whereas $p_2(C)$ is given by $\mu_Q(|C|/n)$, *n* being the cardinality of set *X*.

In addition, it can be demonstrated (Dubois et al., 1988) that this interpretation can also be given by a weighted conjunction:

$$\delta = min_{1 \leq i \leq n} max (1 - w_i, \mu_A(x_i)),$$

where $w_i = 1 - \mu_Q((i-1)/n)$ is the importance given to degree $\mu_A(x_i)$. Here again, *the smaller* $w_i$, *the more neglected* $\mu_A(x_i)$.

This Sugeno fuzzy integral based evaluation is a particular case of a proposition (Bosc & Liétard, 2005) made in a more general framework to evaluate the extent to which an aggregate (computed on a fuzzy set, the cardinality in case of a quantified statement) is confronted to a fuzzy predicate (a linguistic quantifier). So, it can be easily extended to any kind of linguistic quantifiers.

Quantified statements of type "*Q B X* are *A*"

This section presents the previous propositions for the interpretation of fuzzy quantified statements of type "*Q B X* are *A*". Here again, a relative quantifier *Q* is considered.

*Interpretation with an OWA operator*

R.R. Yager (1988) suggests to interpret the expression "*Q B X* are *A*" by an ordered weighted averaging (*OWA*). Let $X = \{x_1, ..., x_n\}$ with: $\mu_B(x_1) \leq \mu_B(x_2) \leq ... \leq \mu_B(x_n)$ and:

$$\sum_{i=1}^{n} \mu_B(x_i) = \text{d}.$$

The weights of the average are defined by:

$$w_i = \mu_Q(S_i) - \mu_Q(S_{i-1}) \text{ with } S_i = \sum_{j=1}^{i} \mu_B(x_j) / d,$$

and $S_0 = 0$. This operator aggregates the values of the implication $\mu_B(x) \rightarrow_{K-D} \mu_A(x)$ where $\rightarrow_{K-D}$ denotes Kleene-Dienes implication ($a \rightarrow_{K-D} b = max\ (1 - a, b)$). If the implication values $c_i$'s are sorted in a decreasing ordrer $c_1 \geq c_2 \geq ... \geq c_n$, the interpretation of "*Q B X* are *A*" is:

$$\sum_{i=1}^{n} (c_i * w_i).$$

This calculus uses an OWA operator to aggregate implication values. As an example, the truth value obtained for "*most of young* employees are *well-paid*" is that of "for *most of* the employees, *to be young* implies *to be well-paid*". The obtained result is far from the original meaning of the quantified statement.

*Interpretation by decomposition*

The interpretation by decomposition described in (Yager 1983, 1984) is limited to increasing quantifiers. The proposition "*Q B X* are *A*" is true if an ordinary subset *C* of *X* satisfies the conditions $p_1$ and $p_2$ given hereafter:

$p_1$: there are *Q* elements *B* in *C*,

$p_2$: each element *x* of *C* satisfies the implication: (*x is B*) $\rightarrow$ (*x is A*).

The truth value of the proposition: "*Q B X* are *A*" is then defined by:

$$sup_{C \in P(X)}\ min\ (p_1(C), p_2(C)),$$

where $p_1(C)$ (resp. $p_2(C)$) denotes the degree of satisfaction of $C$ with respect to the condition $p_1$ (resp. $p_2$). The value $p_1(C)$ is defined by $\mu_Q(h)$ where $h$ is the proportion of elements $B$ in set $C$. Yager suggests the following definition of $h$ (using *∑Counts*):

$$h = \frac{\sum_{x \in C} \mu_B(x)}{\sum_{x \in X} \mu_B(x)}.$$

The value of $p_2(C)$ is:

$$\wedge_{x \in C}\ \mu_B(x) \rightarrow \mu_A(x)$$

where $\wedge$ is any triangular norm and $\rightarrow$ a fuzzy implication.

This interpretation leads to evaluate, the quantified statement by an aggregation of implication values $\mu_B(x) \rightarrow \mu_A(x)$. Similarly to the *OWA* based interpretation of "*Q B X* are *A*", this interpretation is far from the original meaning for "*Q B X* are *A*".

*Proposition of Vila, Cubero, Medina and Pons*

According to this proposition (Vila et al., 1997), the degree of truth for "*Q B X* are *A*" is defined by:

$$\delta = \alpha * max_{x \in X}\ min(\mu_A(x), \mu_B(x)) + (1 - \alpha) * min_{x \in X}\ max(\mu_A(x), 1 - \mu_B(x)),$$

where $\alpha$ is a degree of Orness (Yager & Kacprzyck, 1997) computed from the linguistic quantifier:

$$\alpha = \sum_{i=1}^{n} \left( \frac{(n-i)}{(n-1)} * (\mu_Q(i/n) - \mu_Q((i-1)/n)) \right).$$

The interpretation of "*Q B X* are *A*" is a degree set between the truth value of "$\exists$ *B X* are *A*" (given by $max_{x \in X}\ min(\mu_A(x), \mu_B(x))$) and that of "$\forall$ *B X* are *A*" (given by $min_{x \in X}\ max(\mu_A(x), 1 - \mu_B(x))$). The closer to one is $\alpha$, the more "*Q B X* are *A*" is interpreted as "$\exists$ *B X* are *A*".

*Example.* Let us consider $X = \{x_1, x_2, x_3\}$ where the satisfaction degrees with respect to predicates *B* and *A* are given by table 1.

| | $x_1$ | $x_2$ | $x_3$ |
|---|---|---|---|
| *B* | 1 | 1 | 1 |
| *A* | 1 | 0 | 0 |

**Table 1.** Satisfaction degrees with respect to *B* and *A*

The value of $\alpha$ is given by:

$$\alpha = 1 * (\mu_{almost\ all}(1/3) - \mu_{almost\ all}(0)) + 1/2 * (\mu_{almost\ all}(2/3) - \mu_{almost\ all}(1/3)) + 0 * (\mu_{almost\ all}(1) - \mu_{almost\ all}(2/3)).$$

The linguistic quantifier *almost all* is such that: $\mu_{almost\ all}(0) = 0$, $\mu_{almost\ all}(1/3) = 0.2$, $\mu_{almost\ all}(2/3) = 0.8$ and $\mu_{almost\ all}(1) = 1$ and we get:

$$\alpha = 1 * (0.2 - 0) + 1/2 * (0.8 - 0.2) + 0 * (1 - 0.8) = 0.2 + 0.3 = 0.5.$$

The final result is then:

$$\delta = \alpha * max_{x \in X}\ min(\mu_A(x),\ \mu_B(x)) + (1 - \alpha) * min_{x \in X}\ max(\mu_A(x), 1\text{-}\mu_B(x))$$

$$= 0.5 * 1 + (1\text{-}0.5) * 0 = 0.5.$$

As a consequence "*almost all B X* are *A*" is true at degree *0.5* which is far from the expected result (since the proportion of *A* elements among the *B* elements is *1/3* and $\mu_{almost\ all}(1/3) = 0.2$).

♦

*The GD method for "Q B X are A" statements*

Delgado et al. (Delgado et al., 2000) propose a probabilistic view of the proportion of *A* elements among the *B* elements. Computations are related to the two fuzzy sets *B(X)* and $A(X) \cap B(X)$. In addition, when fuzzy sets *B(X)* and $(A(X) \cap B(X))$ are not normal, they should be normalized (using any technique).

The set $S = \{\alpha_1, \alpha_2, ..., \alpha_m\}$ is the set made of the different satisfaction degrees of elements from *X* with respect to fuzzy conditions *B* and $A \cap B$ (it is considered that $1 = \alpha_1 > \alpha_2 > ... > \alpha_m > \alpha_{m+1} = 0$) and *P* the set of the different proportions provided by the α-cuts :

$$P = \{\frac{|A(X)_\alpha \cap B(X)_\alpha|}{|B(X)_\alpha|} \text{ where } \alpha \text{ is in } S\}.$$

If we denote $P^{-1}(c)$ the set of levels from *S* having *c* as relative cardinality (*c* being in *P*) :

$$P^{-1}(c) = \{\alpha_i \text{ from } S \text{ such that } \frac{|A(X)_{\alpha_i} \cap B(X)_{\alpha_i}|}{|B(X)_{\alpha_i}|} = c\},$$

the probability $p(c)$ for a proportion $c$ (in $[0,1]$) to represent $\frac{|A(X) \cap B(X)|}{|B(X)|}$ is defined by :

$$p(c) = \sum_{\alpha_i \text{ in } P^{-1}(c)} (\alpha_i - \alpha_{i+1}) = \sum_{\alpha_i \text{ such that } c = \frac{|A(X)_{\alpha i} \cap B(X)_{\alpha i}|}{|B(X)_{\alpha i}|}} (\alpha_i - \alpha_{i+1}) .$$

The evaluation of "*Q B X* are *A*" is then :

$$\sum_{c \text{ in } P} p(c) \times \mu_Q(c).$$

About the proposed approaches to evaluate quantified statements

Some properties to be verified by any technique to evaluate quantified statements of type "*Q X* are *A*" and "*Q B X* are *A*" have been proposed in the litterature (Delgado et al., 2000; Blanco et al., 2002) and it is possible to situate the different propositions with respect to this properties. At first, these properties are introduced and then the evaluation of quantified statements are discussed.

Concerning "*Q X* are *A*" statements the following properties can be considered :

Property 1. If predicate A is crisp, the evaluation must deliver $\mu_Q(|A(X)|)$ in case of absolute quantifier and $\mu_Q(|A(X)|/n)$ in case of a relative quantifier (where A(X) is the crisp set made of element from *X* which satisfy *A* and n is the cardinality of crisp set *X*).

Property 2. The evaluation is coherent with the universal and existential quantifiers. It means the evaluation of "*Q X* are *A*" is $\vee_{x \in X} \mu_A(x)$ when *Q* is $\exists$ and $\wedge_{x \in X} \mu_A(x)$ when *Q* is $\forall$ ($\vee$ and $\wedge$ being respectively a conorm and a norm).

Property 3. The evaluation is coherent with quantifiers inclusion. Given two quantifiers *Q* and *Q'* such that $Q \subseteq Q'$ ($\forall x$, $\mu_Q(x) \leq \mu_{Q'}(x)$), the evaluation of "*Q X* are *A*" cannot be larger then that of "*Q' X* are *A*".

Concerning the "*Q B X* are *A*" statements, it is possible to recall:

Property 4. If *A* and *B* are crisp and *Q* is relative, the evaluation must *deliver* $\mu_Q(|A(X) \cap B(X)|/|B(X)|)$ where A(X) (resp. B(X)) is the set made of elements from *X* which satisfy *A* (resp. *B*).

Property 5. When *B* is a Boolean predicate, the evaluation of "*Q B X* are *A*" is similar to that of "*Q* B(X) are *A*" where B(X) is the (crisp) set made of elements from *X* which satisfy *B*.

Property 6. If the set of elements which are *B* is included in the set of *A* elements, *Q* is relative and *B* is normalized, then the evaluation of "*Q B X* are A" is $\mu_Q(1)$ (since 100% of *B* elements are *A* due to the inclusion).

Property 7. If $A(X) \cap B(X) = \varnothing$ (where $A(X)$ (resp. $B(X)$) is the set made of elements from $X$ which satisfy $A$ (resp. $B$)), then the evaluation must return the value $\mu_Q(0)$.

When considering the evaluation of "*Q X* are *A*" statements, the approaches based on cardinalities deliver a result which can be difficult to interpret. In case of a precise cardinality, the main drawback is that a large number of elements with small membership degrees may have same effect on the result than a small number of elements with large membership degrees. As a consequence, property 2 cannot be satisfied (this behaviour is demonstrated in (Delgado et al., 2000)). In addition as shown in (Delgado et al., 2000), property 1 and 3 are satisfied. In case of an imprecise cardinality, the result of the interpretation is imprecise since it takes the form of two indices : a degree of possibility and a degree of necessity. This imprecision tied to the result is difficult to justify because computations take into account a precise quantifier and precise degrees of satisfaction, so why should it delivers an imprecise result ? Moreover, the approaches to evaluate "*Q X* are *A*" using a relaxation mechanism provide a result with a clear meaning and easy to interpret. Theses approaches (including ZS technique) satisfy (Delgado et al., 2000) properties 1, 2 and 3.

When considering the evaluation of "*Q B X* are *A*" statements, the approach based on the *OWA* operator and on a decomposition technique considers a modification of the meaning of the quantified statement since "*Q B X* are *A*" is interpreted as "for *Q* elements in *X*, to satisfy *B* implies to satisfy *A*". These two approaches satisfy properties 4 and 5, while properties 7 and 6 are not fulfilled (Delgado et al., 2000). The approach proposed by Vila et al. (1997) interpret the quantified statement by a compromise between "*∃ B X* are *A*" and " *∀ B X* are *A*". As a consequence, it may leads to a result which does not fit the quantifier's definition (and none of the properties introduced in this section can be satisfied (Delgado et al., 2000)). The method GD satisfies all properties (properties 4, 5, 6, 7).

Next sections show that the framework of gradual numbers offers powerful tools to evaluate quantified statements. This context allows to unify the previous propositions made to evaluate quantified statements of type "*Q X* are *A*" (and based on a relaxation mechanism). In addition, gradual numbers offers new techniques to evaluate "*Q B X* are *A*" statements.

## GRADUAL NUMBERS AND GRADUAL TRUTH VALUE

It has been shown (Rocacher, 2003) that dealing with both quantification and preferences defined by fuzzy sets leads to define gradual natural integers (elements of $\mathbb{N}_f$) corresponding to fuzzy cardinalities. Then, $\mathbb{N}_f$ has been extended to $\mathbb{Z}_f$ (the set of gradual relative integers) and $\mathbb{Q}_f$ (the set of gradual rationals) in order to deal with queries based on difference or division operations (Rocacher & Bosc, 2005). These new frameworks provide arithmetic foundations where difference or ratio between gradual quantities can be evaluated. As a consequence, gradual numbers are essential in particular for dealing with flexible queries using absolute or relative fuzzy quantifiers. This is the reason why this section shortly introduces $\mathbb{N}_f$, the set of gradual integers, and its extensions $\mathbb{Z}_f$ and $\mathbb{Q}_f$. Then, it is shown that applying a fuzzy predicate on a gradual number provides a specific truth value which is also gradual.

Gradual natural integers

The fuzzy cardinality $|F|$ of a fuzzy set $F$, as proposed by Zadeh (1983) is a fuzzy set on $\mathbb{N}$, called *FGCount*($F$), defined by:

$$\forall n \in \mathbb{N},\ \mu_{|F|}(n) = sup\{\alpha \mid |F_\alpha| \geq n\},$$

where $F_\alpha$ denotes an $\alpha$-cut of fuzzy set $F$. The degree $\alpha$ associated with a number $n$ in the fuzzy cardinality $|F|$ is interpreted as the extent to which $F$ has at least $n$ elements. It is a normalized fuzzy set of integers and the associated characteristic function is nonincreasing.

*Example.* The fuzzy cardinality of the fuzzy set $F = \{1/x_1, 1/x_2, 0.8/x_3, 0.6/x_4\}$ is: $|F| = \{1/0, 1/1, 1/2, 0.8/3, 0.6/4\}$. The amount of data in $F$ is completely and exactly described by $\{1/0, 1/1, 1/2, 0.8/3, 0.6/4\}$. Degree *0.8* is the extent to which $F$ contains at least 3 elements.♦

It is very important to notice that we do not interpret a fuzzy cardinality as a fuzzy number based on a possibility distribution (which has a disjunctive interpretation). In fact, the knowledge of all the cardinalities of all different $\alpha$-cuts of a fuzzy set *F* provides an exact characterization of the number of elements belonging to *F*. Consequently $|F|$ must be viewed as a conjunctive fuzzy set of integers. As matter of fact, the considered fuzzy set *F* represents a perfectly known collection of data (without uncertainty), so its cardinality $|F|$ is also perfectly known. We think that it is more convenient to qualify such cardinality as a “gradual” number rather than a “fuzzy” number. Other fuzzy cardinalities based on the definition of FGCounts, such as FLCounts or FECounts, have been defined by Zadeh (1983) or Wygralak (1999). Dubois and Prade (1985) and Delgaldo et al. (2002) have adopted a possibilistic point of view where a fuzzy cardinality is interpreted as a possibility distribution over $\alpha$-cuts corresponding to a fuzzy number (D. Dubois, H. Prade, 1987).

The rest of this paper is based on such a fuzzy cardinality defined as FGCounts and the set of all fuzzy cardinalities is called $\mathbb{N}_f$ (the set of **gradual** natural integers).

The $\alpha$-cut $x_\alpha$ of gradual natural integer *x* is an integer defined as the highest integer value appearing in the description *x* associated with a degree at least equal to $\alpha$. In other words, it is the largest integer appearing in the $\alpha$-level cut of its representation:

$$x_\alpha = max\{c \in \mathbb{N} \mid \mu_x(c) \geq \alpha\}.$$

When x describes the FGCount of a fuzzy set *A*, the following equality holds:

$$x_\alpha = |A_\alpha|.$$

This approach is along the line presented by Dubois and Prade (2005) where they introduce the concept of fuzzy element *e* in a set *S* defined as an assignment function $a_e$ from a complete lattice to *L-{0}* to *S*. Following this view, a gradual natural integer *x* belonging to $\mathbb{N}_f$ can be defined by an assignment function $a_x$ from ]0, 1] to $\mathbb{N}$ such that:

$\forall \alpha \in ]0, 1], a_x(\alpha) = x_\alpha.$

If *x* is identified to a fuzzy cardinality $|F|$ of a fuzzy set *F*, then $a_x(\alpha)$ is the cardinality of the α level cut of *F*.

*Example.* $|F| = \{1/0, 1/1, 1/2, 0.8/3, 0.6/4\}$ is a gradual natural integer defined by an assignment a function $a_{|F|}$ graphically represented by figure 3.

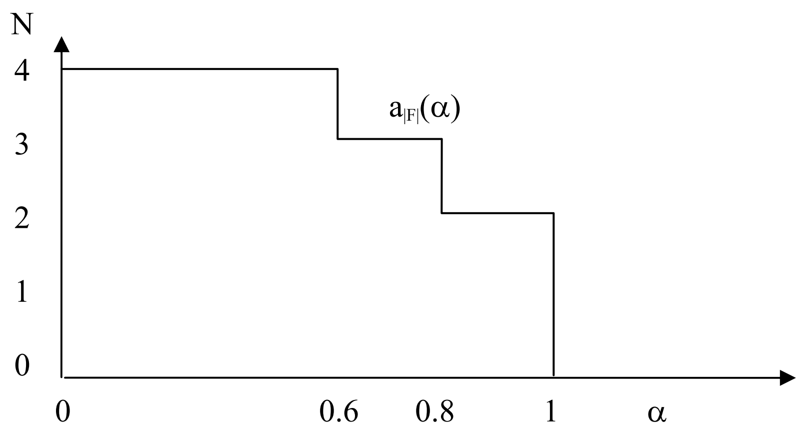


**Figure 3.** The assignment function of a fuzzy cardinality.

As an example, $a_{|F|}(0.7) = |F_{0.7}| = 3$. ♦

Any operation # between two natural integers can then be extended to gradual natural integers *x* and *y* (Rocacher & Bosc, 2005) by defining the corresponding assignment function $a_{x\#y}$ as follows:

$\forall \alpha \in ]0, 1], a_{x\#y}(\alpha) = a_x(\alpha) \# a_y(\alpha) = x_\alpha \# y_\alpha$.

Due to the specific characterization of gradual integers, it can easily be shown that $\mathbb{N}_f$ is a semi-ring structure. So the addition and product operations satisfy the following properties: ($\mathbb{N}_f$, +) is a commutative monoïd (+ is closed and associative) with the neutral element {*1/0*} ; ($\mathbb{N}_f$, ×) is a monoïd with the neutral element {*1/0, 1/1*}; the product is distributive over the addition.

Gradual relative integers

In $\mathbb{N}_f$ the difference between two gradual natural integers may be not defined. As a consequence, $\mathbb{N}_f$ has to be extended to $\mathbb{Z}_f$ in order to build up a group structure.

The set of gradual relative integers $\mathbb{Z}_f$ is defined by the quotient set $(\mathbb{N}_f \times \mathbb{N}_f) / \mathcal{R}$ of all equivalence classes on $(\mathbb{N}_f \times \mathbb{N}_f)$ with regards to $\mathcal{R}$ the equivalence relation characterized by:

$$\forall (x^+, x^-) \in \mathbb{N}_f \times \mathbb{N}_f,\ \forall (y^+, y^-) \in \mathbb{N}_f \times \mathbb{N}_f, (x^+, x^-)\ \mathcal{R}\ (y^+, y^-)\ \text{iff}\ x^+ + y^- = x^- + y^+.$$

The α-cut of a fuzzy relative integer $(x^+, x^-)$ is defined as the relative integer $(x^+_\alpha - x^-_\alpha)$. Any fuzzy relative integer $x$ has a unique canonical representative $x^c$ which can be obtained by enumerating the values of its different α-cuts on ℤ:

$$x^c = \sum \alpha_i / (x^+_{\alpha_i} - x^-_{\alpha_i})$$

where $\alpha_i$'s correspond to the different degrees appearing in the representation of $x^+$ and $x^-$. Each value $x_\alpha$ can be computed from the canonical representation since $x_\alpha$ equals $\mu_{x^c}(\beta)$ with $\beta$ the immediate value larger than or equal to $\alpha$.

The assignment function $a_x$ of $x$ is a function from $]0, 1]$ to ℤ such that:

$$\forall \alpha \in ]0, 1], a_x(\alpha) = x^+_\alpha - x^-_\alpha = x_\alpha$$

*Example.* The compact denotation of the fuzzy relative $(x, y)$ (with: $x = \{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}$ and $y = \{1/0, 1/1, 0.9/2\}$) is:

$$(x, y)^c = \{1/0, 0.9/\text{-}1, 0.8/0, 0.5/1, 0.2/2\}^c.$$

As an example, for a level of $0.9$ we get : $x_{0.9} = 1$ while $y_{0.9} = 2$. As a consequence, the *α-cut* of $(x, y)$ at level $0.9$ is $x_{0.9} - y_{0.9} = -1$. The assignment function of $(x, y)$ is represented by figure 4.

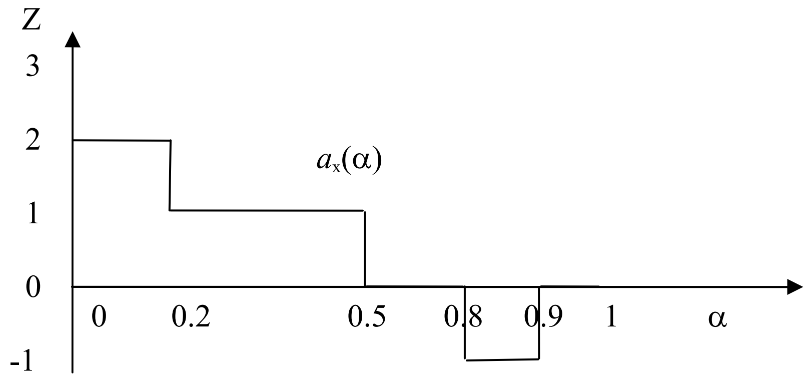


**Figure 4.** Assignment function of the gradual relative integer $(x, y)$♦

If $x$ and $y$ are two gradual relative integers, the addition $x + y$ and the multiplication are respectively defined by the classes $(x^+ + y^+, x^- + y^-)$ and $((x^+ \times y^+) + (x^- \times y^-), (x^+ \times y^-) + (x^- \times y^+))$. The addition is commutative, associative and has a neutral element, denoted by $0_{\mathbb{Z}_f}$, defined by the class $\{(x, x) / x \in \mathbb{N}_f\}$.

Each fuzzy relative integer $(x^+, x^-)$ has an opposite, denoted by $-x = (x^-, x^+)$. This is remarkable because in the framework of usual fuzzy numbers this property is not always satisfied. It can be easily checked that the product in $\mathbb{Z}_f$ is commutative, associative and distributive over the addition. The neutral element is the fuzzy relative integer $(\{1/0, 1/1\}, \{1/0\})$. Therefore we conclude that $(\mathbb{Z}_f, +, \times)$ forms a ring.

Gradual rational numbers

The question is now to define an inverse to each gradual integer and to build up the set of gradual rational numbers. We define $\mathbb{Z}_f^*$ as the set of gradual integer $x$ such that: $\forall\alpha \in ]0, 1], x_\alpha \neq 0$ and $\mathcal{R}'$ as the equivalence relation such that:

$$\forall(x, y) \text{ and } (x', y') \in \mathbb{Z}_f \times \mathbb{Z}_f^*, [x, y] \mathcal{R}' [x', y'] \text{ iff } x \times y' = x' \times y.$$

The set of fuzzy rational numbers $\mathbb{Q}_f$ is defined by the quotient set $(\mathbb{Z}_f \times \mathbb{Z}_f^*) / \mathcal{R}'$.

The representation of a fuzzy relational number $x$ can also be represented thanks to a more simple compact representation (denoted by $x^c$) by enumerating values associated with the different α-cuts which are rationals. The assignment function $a_x$ of $x$ is a function from $]0, 1]$ to $\mathbb{Q}$ is defined by:

$$\forall \alpha \in ]0, 1], a_x(\alpha) = reduce((x^{n+}_\alpha - x^{n-}_\alpha) \div (x^{d+}_\alpha - x^{d-}_\alpha)).$$

where the operator *reduce* means that the rational is reduced to its canonical form.

Gradual truth value

This section proposes a computation to determine the truth value obtained when applying a fuzzy predicate on a gradual number. Let be $x$ an element of $\mathbb{N}_f$ or $\mathbb{Z}_f$ or $\mathbb{Q}_f$, its assignment function $a_x$ is defined by: $\forall \alpha \in ]0, 1], a_x(\alpha) = x_\alpha$. If $T$ is a fuzzy predicate, the application of the predicate *T* on *x* produces a global satisfaction *S* (called gradual truth value) characterized by the assignment function defined by:

$$\forall\, \alpha \in [0, 1],\ a_S(\alpha) = T(x_\alpha) = T(a_x(\alpha)).$$

For a given level $\alpha$, $a_S(\alpha)$ represents the satisfaction of the corresponding α-cut of the fuzzy number. In other words, for a given level $\alpha$, the fuzzy number satisfies predicate *T* at degree $a_S(\alpha)$.

*Example.* We consider the fuzzy predicate *high* defined by figure 5.

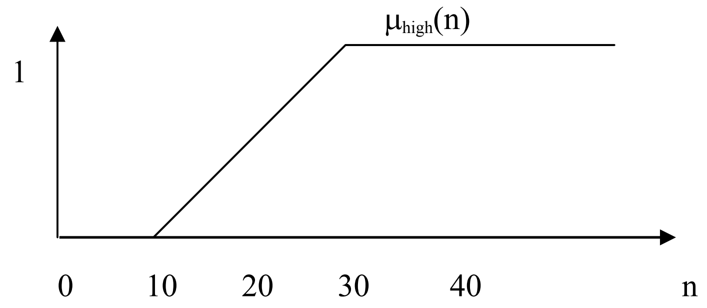


**Figure 5.** The fuzzy predicate *high*.

If the number of young employees is the gradual integer $x = \{1/15,\ 0.7/20,\ 0.2/25\}$ (which means that 15 employees are completely *young*, 5 employees have the same age and are *young* at the level *0.7*, whereas 5 other people are rather not young since their level of youth is estimated at *0.2*). The assignment function for x is the following :

$$\forall\alpha \in [0, 0.2],\ a_x(\alpha) = 25, \qquad \forall\alpha \in ]0.2, 0.7],\ a_x(\alpha) = 20, \qquad \forall\alpha \in ]0.7, 1],\ a_x(\alpha) = 15.$$

The application of the predicate *high* on the gradual integer $x$ produces a global satisfaction $S$ whose function of assignment is defined by :

$\forall \alpha \in [0, 1], a_S(\alpha) = T(x_\alpha) = T(a_x(\alpha))$.

We get the gradual truth value given by figure 6.

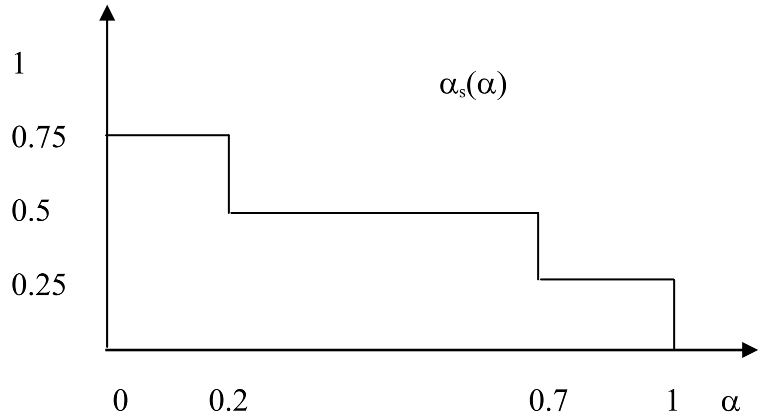


**Figure 6.** Gradual truth value corresponding to a global satisfaction *S* ♦

This gradual truth value shows the different results associated to the different α-cuts. When referring to previous example and when considering level *0.8*, the fuzzy cardinality $x$ states that the cardinality of this α-cut is *15* ($x_{0.8} = 15$). Since $\mu_{almost\ all}(15) = 0.25$, this cardinality satisfies *to be high* at degree *0.25*. It can be checked that $\alpha_s(0.8) = 0.25$.

# INTERPRETATION OF QUANTIFIED STATEMENTS USING GRADUAL NUMBERS

Section "Quantified statements of type "*Q X* are *A*"" considers the evaluation of a quantified statement of type "*Q X* are *A*" while section "Quantified statements of type "*Q B X* are *A*" where *Q* is relative" is interested in the evaluation of statement of type "*Q B X* are *A*", where *Q* is relative. Each one of these computations provides a gradual truth value. As a consequence, section "A scalar truth value for the interpretation" proposes a scalar interpretation computed from this gradual truth value.

## Quantified statements of type "*Q X* are *A*"

The gradual cardinality of the fuzzy set *A*(*X*) made of elements from *X* which satisfy *A* is a FGCount denoted *c* and belongs to $\mathcal{N}_f$. When *Q* is absolute, the gradual truth value for "*Q X* are *A*" is given by the satisfaction of a fuzzy condition (a constraint represented by the quantifier) for that gradual number. As described in section "Gradual truth value", we get:

$$\forall \alpha \in [0, 1],\ \mu_S(\alpha) = \mu_Q(c(\alpha)).$$

From the definition of the FGCount, we get:

$$\forall \alpha \in [0, 1],\ \mu_S(\alpha) = \mu_Q(|A(X)_\alpha|).$$

In other words, the fuzzy truth value *S* expresses the satisfaction of each α-cut of *A*(*X*) with respect to the linguistic quantifier.

In case of a relative linguistic quantifier, the truth value of "*Q X* are *A*" is given by the satisfaction of the linguistic quantifier into the proportion of elements which are *A*. We get:

$\forall \alpha \in [0, 1], \mu_S(\alpha) = \mu_Q(c(\alpha)/n) = \mu_Q(|A(X)_\alpha|/n),$

where *n* is the cardinality of set *X*.

*Example*. We consider the statement "*about 3 X* are *A*" where $X = \{x_1, x_2, x_3, x_4\}$ such that $\mu_A(x_1) = \mu_A(x_2) = 1, \mu_A(x_3) = 0.8, \mu_A(x_4) = 0.6.$ The linguistic quantifier *about 3* is given by figure 7.

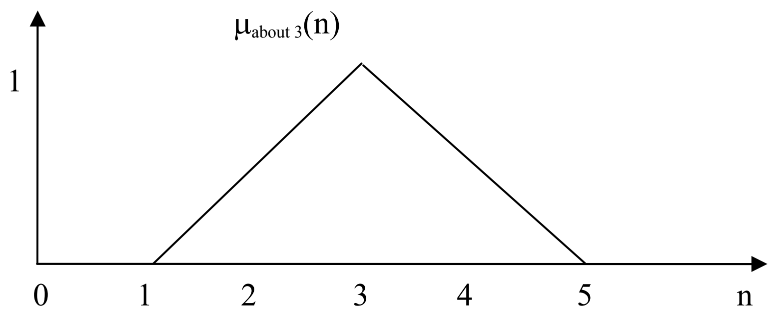


**Figure 7**. A representation for the quantifier *about 3*

The gradual truth value for "*about 3 X* are *A*" (defined by: $\forall \alpha \in [0, 1], \mu_S(\alpha) = \mu_Q(c(\alpha))$) is given by figure 8.

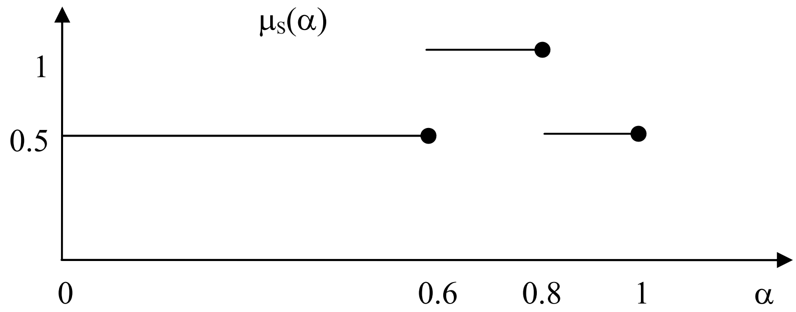


**Figure 8.** A fuzzy truth value for "*about 3 X* are *A*"

This gradual truth value provides the satisfaction obtained for the different α-cuts of *A*(*X*) (set made of elements from *X* which satisfy fuzzy condition *A*). As an example $\mu_S(0.7) = \mu_Q(|A(X)_{0.7}|) = \mu_Q(3) = 1$. ♦

Quantified statements of type "*Q B X* are *A*" where *Q* is relative

The truth value of "*Q B X* are *A*" (*Q* being relative) is given by the satisfaction of the linguistic quantifier into the proportion of elements which are *A* among the elements which satisfy *B*. This proportion is a ration between two gradual integers:

$p = c/d,$

where:

- *c* is the cardinality (FGCount) of the fuzzy set $(A \cap B)(X)$ made of elements from *X* which satisfy fuzzy condition *A* and condition *B* ( $\forall x$ *in X*, $\mu_{A \cap B(X)} = min(\mu_A(x), \mu_B(x)),$

- *d* is the cardinality (FGCount) of the fuzzy set *B*(*X*) made of elements from *X* which satisfy fuzzy condition *B*.

The gradual rational number *c/d* is defined by the couple (*c*, *d*). A canonical representation for *c/n* is:

$\forall \alpha \in [0, 1], p(\alpha) = c(\alpha)/d(\alpha).$

This canonical definition is defined only when $d(\alpha) \neq 0$. The cardinality *c* (resp. *d*) being that of the fuzzy set $(A \cap B)(X)$ ($B(X)$), we get:

$\forall \alpha \in [0, 1], p(\alpha) = |(A \cap B)(X)_\alpha| / |B(X)_\alpha|,$

where $|B(X)_\alpha| \neq 0$. It means that $p(\alpha)$ is not defined for $\alpha > max_{x \in X}\, \mu_B(x)$, and we can write:

$\forall \alpha \in [0, max_{x \in X}\, \mu_B(x)], p(\alpha) = |(A \cap B)(X)_\alpha| / |B(X)_\alpha|.$

The gradual truth value for "*Q B X* are *A*" is given by the satisfaction of the constraint represented by the quantifier for that gradual proportion. According to the results introduced in section "Gradual truth value", a gradual truth value *S* is obtained:

$\forall \alpha \in [0, max_{x \in X}\, \mu_B(x)], \mu_S(\alpha) = \mu_S(p(\alpha)) = \mu_Q(|(A \cap B)(X)_\alpha| / |B(X)_\alpha|).$

The fuzzy truth value *S* expresses the satisfaction of each α-cut of *A(X)* and $(A \cap B)(X)$ with respect to the linguistic quantifier.

The value $\alpha$ is viewed as a quality threshold for the satisfactions with respect to *A* and *B*. When the minimum is chosen as norm to define $(A \cap B)(X)$, the value of $\mu_S(\alpha)$ states that: "among the elements which satisfy *B* at least at level $\alpha$, the proportion of elements *x* with $\mu_A(x) \geq \alpha$, is in agreement with *Q*" (since we have $(A \cap B)(X)_\alpha = A(X)_\alpha \cap B(X)_\alpha$). In other words, $\mu_S(\alpha)$ is the truth value of the quantified statement when considering the two interpretations $A(X)_\alpha$ and $B(X)_\alpha$.

In addition, the fuzzy truth value *S* is not defined when $\alpha > max_{x \in X}\ \mu_B(x)$. A first attitude is to normalize *B* and $A \cap B$ or to employ the degree of *orness* defined by Yager and Kacprzyck (1997) so that that $\mu_S(\alpha) = orness(Q)$. A second attitude which will be considered in this chapter is to assume that $\mu_S(\alpha) = 0$ in that case.

*Example.* We consider the statement "*about half B X* are *A*" where $X = \{x_1, x_2, x_3, x_4\}$. The satisfaction degrees are given by table 2.

| | $x_1$ | $x_2$ | $x_3$ | $x_4$ |
|---|---|---|---|---|
| $\mu_B(x_i)$ | 1 | 0.9 | 0.7 | 0.3 |

| | | | | |
|---|---|---|---|---|
| $\mu_A(x_i)$ | 0.8 | 0.3 | 1 | 1 |
| $\mu_{A \cap B}(x_i)$ | 0.8 | 0.3 | 0.7 | 0.3 |

**Table 2.** Satisfaction with respect to *B* and *A*.

The linguistic quantifier *about half* is given by figure 9.

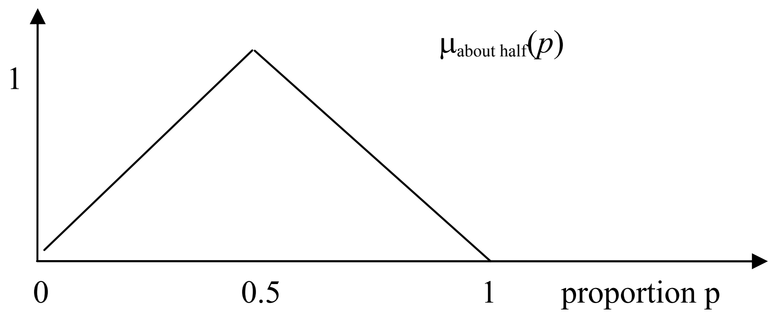


**Figure 9**. A representation for the quantifier *about half*

The gradual truth value for "*about half B X* are *A*"is given by figure 10.

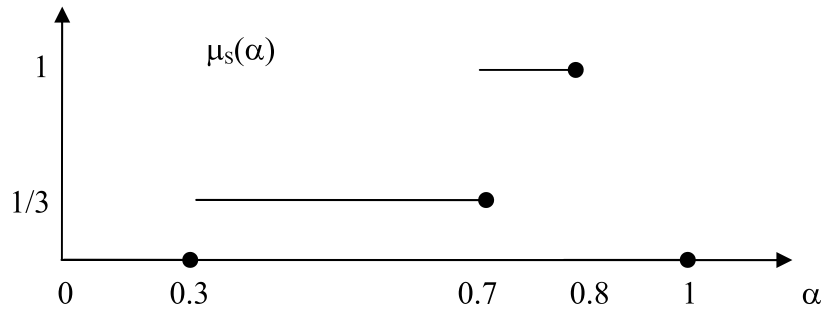


**Figure 10.** A fuzzy truth value for "*about half B X* are *A*"

As an example, we get $\mu_S(0.6) = 1/3$ because $|(A \cap B)(X)_{0.6}|/|B(X)_{0.6}| = 2/3$ and $\mu_Q(2/3) = 1/3$. The truth value of the statement "*about half* elements in $\{x \text{ such that } \mu_B(x) \geq 0.6\}$ are in $\{x \text{ such that } \mu_A(x) \geq 0.6\}$" is *1/3*. ♦

A scalar truth value for the interpretation

The fuzzy truth value *S* computed in the previous section gathers the satisfactions of the different α-cuts with respect to the linguistic quantifier. This fuzzy truth value can be defuzzified in order to obtain a scalar evaluation (set in [0, 1]). Various interpretations can be associated to this defuzzification and we consider the following one (since it is the more natural):

"the more α-cuts highly satisfies the constraint defined by the linguistic quantifier, the higher the scalar interpretation".

Obviously, when the scalar interpretation is *1*, each $\alpha$-cut fully satisfies the constraint. When dealing with a quantified statement of type "*Q X* are *A*", a scalar evaluation of *1* means that whatever is the chosen interpretation for *A*(*X*) (set made of elements from *X* which satisfy *A*), its cardinality is in agreement with the linguistic quantifier (i.e $\forall \alpha, \mu_Q(|A(X)_\alpha|) = 1$ or $\mu_Q(|A(X)_\alpha|/n) = 1$). Otherwise, the higher the scalar evaluation, the more there exists interpretations of *A*(*X*) with a high satisfaction with respect to the linguistic quantifier.

When dealing with a quantified statement of type "*Q B X* are *A*", the scalar evaluation is also interpreted in terms of $\alpha$-cuts, i.e. in terms of interpretations of fuzzy sets. For a given level $\alpha$, the degree $\mu_S(\alpha)$ provided by the gradual truth value represents the satisfaction of the quantifier with respect to the proportion: $|(A \cap B)(X)_\alpha|/|B(X)_\alpha|$ ($\mu_S(\alpha)$ is the truth value of the quantified statement when considering the two interpretations $(A \cap B)(X)_\alpha$ and $B(X)_\alpha$).

The scalar value aggregates the different satisfactions provided by the different levels and a scalar evaluation of 1 means that whatever is the chosen quality threshold $\alpha$, the proportion is in complete agreement with *Q*. Otherwise, the higher the scalar evaluation, the more there exists quality thresholds such that the proportion highly satisfies *Q*.

In section “a quantitative approach”, we consider a quantitative defuzzification (since based on an additive measure – a surface) while in section “a qualitative approach” we consider a qualitative defuzzification (since based on a non additive process). Section “Satisfaction of properties” situates the results provided by these two defuzzifications with respect to the properties introduced in “about the proposed approaches to evaluate quantified statements”.

*A quantitative approach*

In this approach, the surface of the fuzzy truth value is delivered to the user. The scalar interpretation is then (Liétard & Rocacher, 2005):

$$\delta = \left(\int_0^1 \mu_S(\alpha) * p * \alpha^{p-1} d\alpha\right)^{1/p}.$$

When p = *1*, value $\delta$ is the area delimited by function $\mu_S$. Since this function is a stepwise function, we get:

$$\delta = (\alpha_1 - 0) * \mu_S(\alpha_1) + (\alpha_2 - \alpha_1) * \mu_S(\alpha_2) + \ldots + (1 - \alpha_n) * \mu_S(1),$$

where the discontinuity points are: $(\alpha_1, \mu_S(\alpha_1)), \ldots, (\alpha_n, \mu_S(\alpha_n))$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

*Example.* We consider the statement “*about half B X* are *A*” and the fuzzy truth value given by figure 9. We compute:

$$\delta = (0.7 - 0.3) * 1/3 + (0.8 - 0.7)*1 = 0.233.$$

The scalar result is rather low. When referring to table 2, it seems that the proportion of elements which are *A* among the *B* elements is near to be *2/3*. A low result for "*about half* B *X* are *A*" is coherent since the proportion *2/3* poorly satisfies the constraint *about half*. ♦

It has been shown (Liétard & Rocacher, 2005) that, when dealing with quantified statements of type "*Q X* are *A*", this approach is a generalization of the OWA based interpretation (introduced in "Interpretation by the OWA operator"). In addition, next proof shows that when considering "*Q B X* are *A*" statements and when *B* is normalized, this defuzzication leads to the GD method introduced in "The GD method for "Q B X are A" statements" (when *B* is not normalized, the two methods differs since GD method imposes to normalize B, while the gradual truth value associates a value *0* when the α-cuts of B(*X*) is not defined).

**Proof.** In case of a "*Q B X* are *A*" statement, the discontinuity points $(\alpha_i, \mu_S(\alpha_i))$ of the gradual truth value are associated to $\alpha_i$ values where the quantities $\mu_S(\alpha_i)$ vary. In other words:

- $\alpha_i$ values are coming from the set $D = \{\mu_{A \cap B}(x)$ *where x is in X*$\} \cup \{\mu_B(x)$ *where x is in X*$\}$,
- $\mu_S(\alpha_i) = \mu_Q(|(A \cap B)(X)_{\alpha i}| / |B(X)_{\alpha i}|)$.

The defuzzication gives :

$$\delta = (\alpha_1 - 0) * \mu_Q(|(A \cap B)(X)_{\alpha 1}| / |B(X)_{\alpha 1}|) + (\alpha_2 - \alpha_1) * \mu_Q(|(A \cap B)(X)_{\alpha 2}| / |B(X)_{\alpha 2}|)$$

$$+ \ldots + (\alpha_n - \alpha_{n-1}) * \mu_S(1),$$

where the values from D are denoted $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. This expression is clearly that of an interpretation using the GD method (cf. 3.2.4).

*A qualitative approach*

According to this defuzzification, the scalar interpretation takes into consideration two aspects:

- a garanteed (minimal) satisfaction value $\beta$ associated to the α-cuts ($\beta$ must be higher as possible),
- the repartition of $\beta$ among the α-cuts ($\beta$ should be attained by the most possible α-cuts).

Obviously, these two aspects are in opposition since, in general, the higher $\beta$, the smaller the repartition. The scalar interpretation δ reflects a compromise between these two aspects and we get:

$$\delta = \max_{\beta \text{ in } [0,1]} \min(\beta, each(\beta)),$$

where *each*(β) means "*for each level* $\alpha$, $\mu_S(\alpha) \geq \beta$".

A definition of *each*(β) delivering a degree is the more convenient (Bosc & Liétard, 2005) and we propose to sum the lengths of intervals (of levels) where the threshold β is reached:

$$each(\beta) = \sum_{\substack{]\alpha_i, \alpha_j] \text{ such that} \\ \forall \alpha \in ]\alpha_i, \alpha_j],\ \mu_S(\alpha) \geq \beta}} (\alpha_j - \alpha_i).$$

The higher *each*(β), the more numerous the levels $\alpha$ for which $\mu_S(\alpha) \geq \beta$. In particular, *each*(β) equals 1 means that for each level $\alpha$, $\mu_S(\alpha)$ is larger than (or equal to) $\beta$.

In addition, from a computational point of view, the definition of $\delta$ needs to handle an infinity of values $\beta$. However, it is possible (Bosc & Liétard, 2005) to restrict computations to $\beta$ values belonging to the set of "effective" $\mu_S(\alpha)$ values:

$$\delta = \max_{\{\beta \mid \exists \alpha \text{ such that } \beta = \mu_S(\alpha)\}} \min(\beta, each(\beta)),$$

*Example.* We consider the statement "*about half B X* are *A*" and the fuzzy truth value given by figure 9. The values $\beta$ to be considered are *1/3* and *1*. Furthermore:

*each*(*1/3*) = *0.5,*

*each*(*1*) = *0.1.*

We get $\delta = max\ (min(1/3,\ 0.5),\ min(1,\ 0.1)) = 1/3$. As in the previous example, a low result for "*about half B X* are *A*" is coherent. ♦

It has been shown (Bosc & Liétard, 2005), when dealing with quantified statements of type "*Q X* are *A*" (*Q* increasing), this defuzzification leads to the Sugeno fuzzy integral based approach introduced in section "the probabilistic approach (GD method)".

*Satisfaction of properties*

This section situates the results provided by these two defuzzifications with respect to the properties introduced in "about the proposed approaches to evaluate quantified statements". All the properties are satisfied, except property 7 which holds only when the set made of element from *X* which satisfy *B* is normalized.

We recall that the quantitative approach delivers :

$$\delta = (\alpha_1 - 0) * \mu_S(\alpha_1) + (\alpha_2 - \alpha_1) * \mu_S(\alpha_2) + \ldots + (1 - \alpha_n) * \mu_S(1),$$

while the qualitative approach delivers :

$$\delta = max_{\{\beta \mid \exists\, \alpha \text{ such that } \beta = \mu_S(\alpha)\}}\ min(\beta,\ each(\beta)),$$

where *each*(β) is defined by :

$$each(\beta) = \sum_{\substack{]\alpha_i, \alpha_j]\ \text{such that} \\ \forall \alpha \in ]\alpha_i, \alpha_j],\ \mu_S(\alpha) \geq \beta}} (\alpha_j - \alpha_i).$$

and the discontinuity points of the gradual truth value are: $(\alpha_1, \mu_S(\alpha_1)),\ldots, (\alpha_n, \mu_S(\alpha_n))$ with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$.

We first demonstrate the validity of properties related to "*Q X* are *A*" statements, then that of properties related to "*Q B X* are *A*" statements.

*Properties related to "Q X are A" statements*

In case of a " *Q X* are *A* " statement, the $\alpha_i$ are coming from the set $D = \{\mu_A(x)$ *where* $x$ *is in* $X\}$ and $\mu_S(\alpha_i) = \mu_Q(|A(X)_{\alpha_i}|)$ (or $\mu_Q(|A(X)_{\alpha_i}|/n)$ in case of a relative quantifier with $n$ the cardinality of set $X$). The different properties to be satisfied are :

<u>Property 1</u>. If predicate $A$ is crisp, the evaluation must deliver $\mu_Q(|A(X)|)$ in case of absolute quantifier and $\mu_Q(|A(X)|/n)$ in case of a relative quantifier (where A(X) is the crisp set made of element from X which satisfy $A$ and n is the cardinality of crisp set $X$).

**Proof.** When $A$ is crisp, $D$ is a singleton ($\{1\}$) and the only discontinuity point of the gradual truth value (cf. figure 11) is $(1, \mu_Q(|A(X)|)$ (or $(1,\mu_Q(|A(X)|/n)$).


**Figure 11.** The gradual truth value associated to property 1.

The quantitative approach delivers : $(1 - 0) * \mu_Q(|A(X)|)$ (or $(1\text{-}0) * \mu_Q(|A(X)|/n)$ when $Q$ is relative) and property 1 holds. Concerning the qualitative approach, we demonstrate the validity of property 1 only the case of an absolute quantifier. In case of a relative quantifier it is necessary to change each expression $\mu_Q(|A(X)|)$ into $\mu_Q(|A(X)|/n)$ and the demonstration remains valid.

When dealing with the qualitative approach, there is only one value $\beta$ to be considered. This value equals $\mu_Q(|A(X)|)$ and $each(\beta)$ = 1 (since for every level α in [0,1] $\mu_S(\alpha) = \mu_Q(|A(X)_\alpha|) = \mu_Q(|A(X)|) = \beta$). As a consequence the result of the qualitative approach is $min(\beta, each(\beta)) = min(\mu_Q(|A(X)|), 1)$ and the property is valid.

Property 2. The evaluation is coherent with the universal and existential quantifiers. It means the evaluation of "*Q X* are *A*" is $\vee_{x \in X} \mu_A(x)$ when *Q* is $\exists$ and $\wedge_{x \in X} \mu_A(x)$ when *Q* is $\forall$ ($\vee$ and $\wedge$ being respectively a conorm and a norm).

**Proof.** The universal quantifier is relative and defined by $\mu_\forall(1) = 1$ and for any *k* in $[0,1[$, $\mu_\forall(k) = 0$. The gradual truth value is defined by:

- $\mu_S(\alpha) = \mu_\forall(|A(X)_\alpha|/n) = 1$ when $|A(X)_\alpha|/n = 1$ which means when $\alpha$ is smaller than the minimum of membership degrees (denoted $\alpha_1$). This value $\alpha_1$ can be equal to $0$ (when there exists at least one element x with $\mu_A(x) = 0$).
- $\mu_S(\alpha) = 0$ otherwise.

As a consequence, we obtain the gradual truth value given by figure 12.

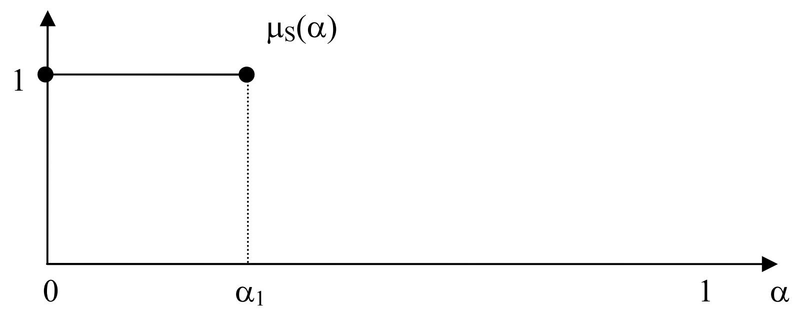


**Figure 12.** The gradual truth value associated to the universal quantifier.

The fuzzy truth value has a unique discontinuity point $(\alpha_1, 1)$ and the quantitative approach delivers $\delta = (\alpha_1 - 0) * 1 = \alpha_1$ which is the minimum of the membership degrees. Property 2 is then satisfied using the minimum as a norm. When dealing with the qualitative approach, there is only one value $\beta$ to be considered. This value equals $\beta = 1$ with $each(\beta) = \alpha_1$ . As a consequence the result is $min(\beta, each(\beta)) = \alpha_1$ and the property is valid.

The existential quantifier is absolute and defined by $\mu_\exists(0) = 0$ and for any $k \neq 0$, $\mu_\exists(k) = 1$. The discontinuity points of the gradual truth value are: $(\alpha_1, 1), ..., (\alpha_n, 1)$ (see figure 13), where $\alpha_n$ is the highest degree among the $\mu_A(x)$s.

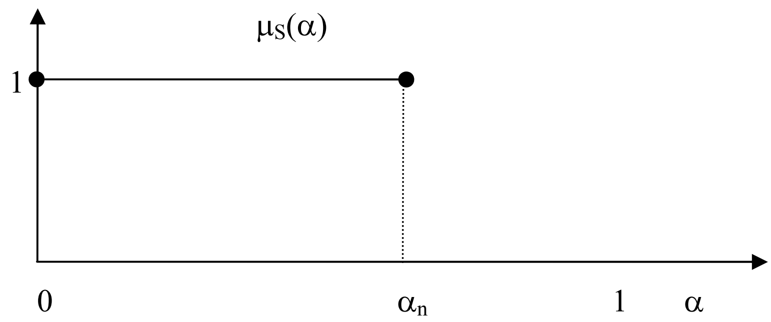


**Figure 13.** The gradual truth value corresponding to the existential quantifier.

The quantitative approach delivers : $\delta = (\alpha_1 - 0) * 1 + (\alpha_2 - \alpha_1) * 1 +...+ (\alpha_n - \alpha_{n-1}) * 1 + (1 - \alpha_n) * 0 = \alpha_n$ which is the maximum of the membership degrees. Property 2 is then satisfied using the maximum as a conorm. When dealing with the qualitative approach, there is only one value $\beta$ to be considered. This value equals $\beta = 1$ with $each(\beta) = \alpha_n$ . As a consequence the result is $min(\beta, each(\beta)) = \alpha_n$ and the property is valid.

Property 3. The evaluation is coherent with quantifiers inclusion. Given two quantifiers $Q$ and $Q'$ such that $Q \subseteq Q'$ ($\forall x,\ \mu_Q(x) \leq \mu_{Q'}(x)$), the evaluation of "*Q X* are *A*" cannot be larger then that of "*Q' X* are *A*".

**Proof.** The gradual truth value for "*Q X* are *A*" is denoted *S*, while that associated to "*Q' X* are *A*" is denoted *S'*. Since we have : $\forall x,\ \mu_Q(x) \leq \mu_{Q'}(x)$, it implies $\forall\, \alpha$ in $[0, 1]$ , $\mu_S(\alpha) \leq \mu_{S'}(\alpha)$ (since the two quantified statements are dealing with the same set *X* and the same fuzzy predicate *A*).

We denote $\delta$ and $\delta'$ the respective evaluation of "*Q X* are *A*" and "*Q' X* are *A*". If the quantitative approach is chosen, we have $\delta = \int_0^1 \mu_S(\alpha)d\alpha$ and $\delta' = \int_0^1 \mu_{S'}(\alpha)d\alpha$ . As a consequence, $\delta \leq \delta'$ and property 3 is valid.

If the qualitative approach is chosen :

$\delta = max_{\beta \, in \, [0,1]} \, min(\beta, each(\beta))$, with $each(\beta) = \sum_{\substack{]\alpha_i, \alpha_j] \text{ such that} \\ \forall \alpha \in ]\alpha_i, \alpha_j], \ \mu_S(\alpha) \geq \beta}} (\alpha_j - \alpha_i)$

$\delta' = max_{\beta \, in \, [0,1]} \, min(\beta, each'(\beta))$, with $each'(\beta) = \sum_{\substack{]\alpha_i, \alpha_j] \text{ such that} \\ \forall \alpha \in ]\alpha_i, \alpha_j], \ \mu_{S'}(\alpha) \geq \beta}} (\alpha_j - \alpha_i)$

Since $\forall \alpha$ in $[0, 1]$ , $\mu_S(\alpha) \leq \mu_{S'}(\alpha)$, we have $each(\beta) \leq each'(\beta)$ which gives $\delta \leq \delta'$ and property 3 is demonstrated.

*Properties related to "Q B X are A" statements*

In case of a "*Q B X* are *A*" statement, the discontinuity points $(\alpha_i, \mu_S(\alpha_i))$ of the gradual truth value are associated to $\alpha_i$ values where the quantities $\mu_S(\alpha_i)$ varies. In other words :

- $\alpha_i$ values are coming from the set $D = \{\mu_{A \cap B}(x) \text{ where } x \text{ is in } X\} \cup \{\mu_B(x) \text{ where } x \text{ is in } X\}$,
- $\mu_S(\alpha_i) = \mu_Q(|(A \cap B)(X)_{\alpha i}| / |B(X)_{\alpha i}|)$.

The different properties to be satisfied are:

Property 4. If $A$ and $B$ are crisp and $Q$ is relative, the evaluation must deliver $\mu_Q(|A(X) \cap B(X)|/|B(X)|)$ where $A(X)$ (resp. $B(X)$) is the set made of elements from $X$ which satisfy $A$ (resp. $B$).

**Proof.** When $A$ and $B$ are is crisp, $D$ is a singleton $(\{1\})$ and the only discontinuity point of the gradual truth value is $(1, \mu_Q(|(A \cap B)(X)| / |B(X)|))$ where $(A \cap B)(X)$ and $B(X)$ are crisp sets (see figure 14).

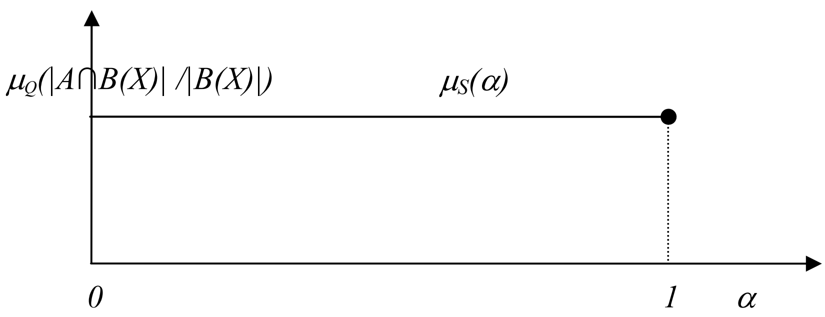


**Figure 14.** The gradual truth value associated to property 4.

The quantitative approach delivers : $(1 - 0) * \mu_Q(|(A \cap B)(X)| / |B(X)|)$ and property 1 holds.

When dealing with the qualitative approach, there is only one value $\beta$ to be considered. This value $\beta$ equals $\mu_Q(|(A \cap B)(X)| / |B(X)|)$ and $each(\beta) = 1$. As a consequence the result of the qualitative approach is $min(\beta, each(\beta)) = min(\mu_Q(|(A \cap B)(X)| / |B(X)|), 1)$ and the property is valid.

Property 5. When *B* is a Boolean predicate, the evaluation of "*Q B X* are *A*" is similar to that of "*Q B*(*X*) are *A*" where B(X) is the (crisp) set made of elements from *X* which satisfy *B*.

**Proof.** This proof shows that the gradual truth value S associated to "*Q B X* are *A*" and the gradual truth value *S'* associated to "*Q* B(X) are *A*" are exactly the same : $\forall \alpha$ in $[0,1]$, $\mu_S(\alpha) = \mu_{S'}(\alpha)$.

When *B* is a Boolean predicate $B(X)_\alpha$ is the crisp set B(X) for any level α. As a consequence, $\mu_S(\alpha) = \mu_Q(|(A \cap B)(X)_\alpha| / |B(X)|)$. Since $(A \cap B)(X)_\alpha$ can be rewritten $A(X)_\alpha \cap B(X)$ we have :

$$\mu_S(\alpha) = \mu_Q(A(X)_\alpha \cap B(X)/|B(X)|).$$

It means that $\mu_S(\alpha)$ is restricted to elements belonging to B(X) as it is the case for "*Q* B(X) are *A*" statement and we obviously get : $\forall \alpha$ in $[0,1]$, $\mu_S(\alpha) = \mu_{S'}(\alpha)$.

Property 6. If the set of elements which are *B* is included in the set of *A* elements, *Q* is relative and *B* is normalized, then the evaluation of "*Q B X* are *A*" is $\mu_Q(1)$ (since 100% of *B* elements are *A* due to the inclusion).

**Proof.** When the set of elements which are *B* is included in the set of *A* elements, we have $\mu_B(x) \leq \mu_A(x)$ for any element *x* from *X*. As a consequence, $B(X)_\alpha \subseteq A(X)_\alpha$ for any level $\alpha$. As a consequence, $\forall\alpha$ in $[0, max_{x \in X}\ \mu_B(x)]$, $\mu_S(\alpha) = \mu_Q(1)$. Since *B(X)* is normalized, $\forall\alpha$ in $[0,1]$, $\mu_S(\alpha) = \mu_Q(1)$ and it is obvious to show that the two defuzzifications give $\mu_Q(1)$ as final results.

Property 7. If $A(X) \cap B(X) = \varnothing$ (where *A(X)* (resp. *B(X)*) is the set made of elements from *X* which satisfy *A* (resp. *B*)), then the evaluation must return the value $\mu_Q(0)$.

We show this property holds only when *B(X)* is normalized.

**Proof.** When $A(X) \cap B(X) = \varnothing$, we have $A \cap B(X)_\alpha = \varnothing$ for any level $\alpha$ in $[0,1]$. As a consequence, $\forall\alpha$ in $[0, max_{x \in X}\ \mu_B(x)]$, $\mu_S(\alpha) = \mu_Q(0)$. When *B(X)* is normalized fuzzy set, we get $\forall\alpha$ in $[0,1]$, $\mu_S(\alpha) = \mu_Q(0)$ and it is obvious to show that the two defuzzifications give $\mu_Q(0)$ as final results.

## CONCLUSION

This paper takes place at the crossword of quantified statements evaluation and fuzzy arithmetic introduced in (Rocacher & Bosc, 2003a, 2003b, 2003c, 2005). It shows that fuzzy arithmetic allows to evaluate quantified statements of type "*Q X* are *A*" and "*Q B X* are *A*".

The evaluation can be either a fuzzy truth value or a scalar value obtained by the defuzzification of the fuzzy value. Two types of scalar values can be distinguished: the first one corresponds to a quantitative view of the fuzzy value, the second one of a qualitative view.

When dealing with quantified statements of type "*Q X* are *A*", the two scalar values are respectively generalizations of the OWA based interpretation and the Sugeno integral based interpretation. When dealing with "*Q B X* are *A*" statements, our approach presents the advantage of providing a theoretical framework for computation. It is the first attempt to set this evaluation in the framework of an extended arithmetic and algebra.

This aspect is very important since properties provided by the algebraic framework hold and we expect to obtain more interesting properties for the qualitative and quantitative approaches (in addition to the ones already stated in this paper). As a consequence, further studies may concern the comparison of the qualitative and quantitative approach in terms of properties. In addition, since they are both summaries of the same evaluation (in the form of a gradual number), they should not differ significantly.

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*Key terms and their definition :*

Fuzzy predicate : Predicate defined by a fuzzy set. A fuzzy predicate delivers a degree of satisfaction.

Gradual integer : Integer which takes the form of a fuzzy subset of the set of naturals (interpreted as a conjunction). Such integers differ from fuzzy numbers which are interpreted as disjunctions of candidates.

Gradual relative integer : Gradual number represented by a fuzzy subset of the set of relatives. (interpreted as a conjunction). It is defined as the substraction of two gradual integers.

Gradual relational number : Gradual number interpreted as a conjunction and defined as the ratio of two relative integers.

Linguistic quantifiers : Quantifiers defined by linguistic expressions like "*around 5*" or "*most of*". Such quantifiers allow an intermediate attitude between the conjunction (expressed by the universal quantifier ∀) and the disjunction (expressed by the existential quantifier ∃).

OWA operator : Ordered Weighted Average Operator. The inputs are assumed to be sorted and the weights of this average are associated to input data depending on their rank (weight $w_1$ is associated to the largest input, weight $w_2$ is associated to the second largest input,…).

Sugeno fuzzy integral : Aggregate operator which can be viewed as a compromise between two aspects : i) a certain quantity (a fuzzy measure) and ii) a quality of information (a fuzzy set).