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The set of fuzzy rational numbers and flexible querying

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Abstract

A fuzzy bag is a bag in which each occurrence of an element is associated with a grade of membership. This notion can be viewed as a generalization of the concepts of set, fuzzy set and bag. The set of fuzzy integers ($\mathbb{N}_f$) provides a general characterization in which all these different concepts are treated in a uniform way and can then be composed. In the field of databases, the use of fuzzy bags is motivated by their ability to manage both quantities and preferences. However, $\mathbb{N}_f$ becomes too restricted a framework when dealing with queries based on difference or division operations. So, a more general structure based on the set of fuzzy relative integers ($\mathbb{Z}_f$) in which exact differences can be performed, has been first developed. In this paper, we carry on with this approach and we extend $\mathbb{Z}_f$ to the set of fuzzy rational numbers ($\mathbb{Q}_f$). This context leads to define a closed system of multiplicative operations and allows to perform exact divisions. Applied to flexible querying of databases, $\mathbb{Q}_f$ and the notion of division on fuzzy numbers allow to generalize the relational division. They define a sound basis for queries involving ratios between quantities.

Keywords: Fuzzy integer; Fuzzy cardinality; Fuzzy rational number; Fuzzy bags; Bags; Fuzzy sets; Flexible querying

1. Introduction

An issue in extending database management functionalities is to increase the expressiveness of query languages. Flexible querying [3] enables users to express preferences inside requirements. Fuzzy set theory offers a general framework for dealing with flexible queries and priorities inside compound queries. The answers to such queries are then qualified and rank-ordered. Besides, the bag type [1,2], which offers

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the capability of managing quantities (numbers of occurrences of data items), plays an important role in databases [15,18] and data models (relational or object oriented) have been designed to support it. Systems taking into account both flexible queries and bags motivate the use of fuzzy bags. For example, a fuzzy bag can be obtained when some attributes are removed from a fuzzy set of tuples. This is illustrated by the query: *find the salaries of young employees* which requires a projection (salary) of a fuzzy set of persons (the young employees) and delivers a fuzzy bag. As several employees may have the same salary, the collection of salaries returned may contain duplicates. Moreover, a given salary occurrence is associated with a more or less young employee and thus satisfies more or less the criterion “to be the salary of a young employee”. Consequently, the different salaries returned by the query have to be managed both quantitatively and qualitatively thanks to a fuzzy bag which represents the distribution of the salaries of young employees.

Our research aim is to devise new structures capable of dealing with quantification and preferences on data. These models can then be used for extending elementary query operators that provide a sound basis for designing high level query languages such as OQL or SQL. So, we are mainly concerned with the study of flexible querying of databases and we follow a pragmatic, application domain-driven approach. But, it is worth mentioning that our investigations have a larger scope than the field of databases and many other potential application domains could also benefit from fuzzy bags, such as fuzzy data mining, summarization of data or fuzzy information retrieval.

Fuzzy bags and some of their operators have been defined by Yager in [30,31] and complementary studies have been carried out in [7,8,9,17,20,21]. In [23,27], we have proposed a new approach for building fuzzy bags so as to introduce operators compatible with both bags and fuzzy sets. Hence, we have shown that fuzzy bags can be viewed as a generalization of fuzzy sets thanks to the consideration of an order structure over the unit interval. Their characteristic function is then defined from a universe U to the set of conjunctive fuzzy natural integers ($\mathbb{N}_f$). However, in this context, the difference operation between two bags A and B cannot always be computed. This problem comes from the fact that the fuzzy bag model considered so far is based on positive fuzzy integers. It is the reason why the set of fuzzy relative integers ($\mathbb{Z}_f$) was constructed. In such a framework, as discussed in [25], the difference $A - B$ of two fuzzy bags is always defined.

This paper, situated in the continuation of these works, aims at extending $\mathbb{Z}_f$ to $\mathbb{Q}_f$, the set of fuzzy rational numbers. This context leads to define a closed system of multiplicative operations and to perform exact divisions. The role of these arithmetic structures is illustrated in the field of flexible querying of databases where $\mathbb{Q}_f$ and the notion of division on fuzzy numbers allow to generalize the relational division or to define a sound basis for queries calling on ratios between quantities.

The rest of this paper is organized as follows. Sections 2 and 3 recall some key notions which constitute the background of the new contributions developed in Sections 4 and 5. Thus, Section 2 introduces the concepts of fuzzy bags and fuzzy natural integers. The main definitions and operators are recalled in Subsections 2.1 and 2.2. In Section 3, the extension of $\mathbb{N}_f$ to $\mathbb{Z}_f$ and the concept of a fuzzy bag defined on $\mathbb{Z}_f$ are briefly discussed. Next, Section 4 is devoted to a complementary study extending $\mathbb{Z}_f$ to $\mathbb{Q}_f$. Main definitions, operators and algebraic properties are first analyzed, then the exact division on $\mathbb{Q}_f$ and its different approximations on $\mathbb{N}_f$ or $\mathbb{R}$ are more specifically considered. Last, the usefulness of these propositions is emphasized in the database domain. Thus, in Section 5, we first study some generalizations of relational applications thanks to approximate divisions, then we illustrate the interest of $\mathbb{Q}_f$ when dealing with a query such as *what is the average salary of a fuzzy set of young employees*? Or when evaluating grades of inclusion and similarity measures based on divisions of fuzzy cardinalities.

2. Fuzzy bags and fuzzy natural integers

In this section, we show how to specify fuzzy bags thanks to the concept of fuzzy natural integer so as to introduce a structure compatible with both bags and fuzzy sets.

2.1. Fuzzy bags characterizations

A fuzzy bag is a bag in which each occurrence of an element is associated with a grade of membership [30]. One way to describe a fuzzy bag is to enumerate its elements, for example: $A = \langle 1/a, 0.1/a, 0.1/a, 0.5/b\rangle$.

Thus, a fuzzy bag is a collection of elements with multiple occurrences and having degrees of membership. Bags and fuzzy sets can be viewed as particular cases of fuzzy bags. In the following, taking into account this duality, we show that as the concept of α-cuts can be viewed as a bridge connecting bags and fuzzy bags, symmetrically, the concept of ω-cuts (which is similar to α-cuts but related to numbers of occurrences) established a link between fuzzy sets and fuzzy bags [23].

Bag operators can be extended to fuzzy bags thanks to the α-cut concept, similarly to the extension of a set into a fuzzy set. We define the α-cut of a fuzzy bag A as the crisp bag A_α which contains all the occurrences of the elements of a universe U whose grade of membership in A is greater than (or equal) to the degree α ($\alpha \in]0, 1]$). The number of occurrences of the element x in A_α is denoted by: $\omega_{A_\alpha}(x)$. In order to preserve the compatibility between bag and fuzzy bag structures, we define the intersection and the union of fuzzy bags satisfying the following properties:

$$(A \cap B)_\alpha = A_\alpha \cap B_\alpha; \quad (A \cup B)_\alpha = A_\alpha \cup B_\alpha, \tag{1}$$

where the union and intersection on bags A_α, B_α are characterized by $\omega_{A_\alpha \cap B_\alpha}(x) = \min(\omega_{A_\alpha}(x), \omega_{B_\alpha}(x))$, $\omega_{A_\alpha \cup B_\alpha}(x) = \max(\omega_{A_\alpha}(x),\ \omega_{B_\alpha}(x))$.

Symmetrically, we bind fuzzy bag and fuzzy set structures by introducing the concept of ω-cut. The ω-cut of a fuzzy bag A is the fuzzy set A^ω such that the grade of membership of the element x in A^ω, denoted by $\mu_{A^\omega}(x)$, defines the extent to which A contains *at least* ω (with $\omega \in \mathbb{N}^+$) occurrences of x:

$$\mu_{A^\omega}(x) = \sup\{\alpha | \omega_{A\alpha}(x) \geqslant \omega\}. \tag{2}$$

Such a function allows to extend operations on fuzzy sets to their counterparts on fuzzy bags thanks to the following properties:

$$(A \cap B)^\omega = A^\omega \cap B^\omega; \quad (A \cup B)^\omega = A^\omega \cup B^\omega, \tag{3}$$

where the union and intersection on fuzzy sets A^ω, B^ω are characterized by $\mu_{A^\omega \cap B^\omega}(x) = \min(\mu_{A^\omega}(x), \mu_{B^\omega}(x))$, $\mu_{A^\omega \cup B^\omega}(x) = \max(\mu_{A^\omega}(x), \mu_{B^\omega}(x))$.

These two characterizations lead us to put forward a new approach which merges both degrees and numbers of occurrences into the unique concept of fuzzy natural integer.

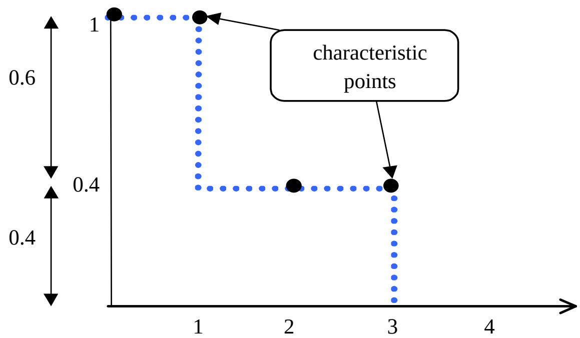

Fig. 1. A graphical representation of FGCount$(A) = \{1/0, 1/1, 0.4/2, 0.4/3\}$.

2.2. Fuzzy natural integers

The fuzzy cardinality $|A|$ of a fuzzy set A, as proposed by Zadeh [31], is called FGCount(A) and defined by

$$\forall n \in N, \quad \mu_{|A|}(n) = \sup\{\alpha | |A_\alpha| \geqslant n\}. \tag{4}$$

Let us consider the fuzzy set $A = \{1/x_1, 0.4/x_2, 0.4/x_3\}$, then the fuzzy cardinality of A is $|A| = \{1/0, 1/1, 0.4/2, 0.4/3\}$. The degree α associated with a number ω in the fuzzy cardinality $|A|$ is interpreted as the extent to which A has *at least* ω elements. It is a normalized convex fuzzy set of integers and the associated characteristic function is nonincreasing.

The amount of data in A is completely and exactly described by the fuzzy set $\{1/0, 1/1, 0.4/2, 0.4/3\}$ (a graphical representation of $|A|$ is given by Fig. 1). But, other denotations may be used for representing $|A|$. For example, a possibilistic view of $|A|$ is given by the fuzzy set of the cardinalities of all its α-cuts: $\{1/1, 0.4/3\}_2$. Such a "compact" representation of the cardinality of A is only specified by the characteristic points $(1, 1)$ and $(3, 0.4)$. Fig. 1 can also be described using a probabilistic notation given by $\{0.6/1, 0.4/3\}_3$. So, we can view $\{1/0, 1/1, 0.4/2, 0.4/3\}$, $\{1/1, 0.4/3\}_2$, $\{0.6/1, 0.4/3\}_3$ as three different descriptions of the *same* information. Whatever the used representation, operations on fuzzy cardinalities have to produce equivalent results. The advantage of the first one is its convexity. Consequently, it satisfies the additivity property of classical cardinalities (based on the extension principle) [11].

It is very important to notice that this kind of "fuzzy number" is not interpreted as a possibility distribution, as "usual fuzzy numbers" [11] are, but it is viewed as a *conjunctive* fuzzy set of integers. In fact, the knowledge of all the cardinalities of all the different α-cuts of a fuzzy set A provides an exact characterization of the number of elements belonging to A. Of course, the considered fuzzy set A represents a perfectly known collection of data (without uncertainty), consequently its cardinality is also perfectly known. We think that it is more convenient to qualify such a cardinality number as being "gradual" rather than being "fuzzy". As shown in the following, this specificity has important consequences regarding the validity of group properties (in a mathematical meaning) which hold in this particular context.

Using definition (4) the cardinality of a crisp set E is an increasing set of integers $\{0, 1, \ldots, n\}$. Such a set represents a cardinality and is also mathematically equivalent to the integer n. This approach is conformed to the classical mathematical definition calling n the cardinality of a crisp finite set E when there exists a bijection between E and an increasing set of integers $\{1, \ldots, n\}$.

The cardinality of an α-cut of a fuzzy set E is the corresponding α-cut of its fuzzy cardinality $\{0, 1, \ldots, n\}$ assimilated to the integer n. Thus, a fuzzy cardinality, such as $\{1/0, 1/1, 0.4/2, 0.4/3\}$, can be viewed as a fuzzy integer, and from now on, the set of all fuzzy cardinalities (defined as FG-Counts) will be called $\mathbb{N}_\mathrm{f}$ (the set of *fuzzy* natural integers). It is important to notice that each α-cut of a fuzzy integer, as considered here, is viewed as an integer. On the contrary, an α-cut of a "usual fuzzy number", interpreted as an ill-known number, represents various possible values (i.e. a *disjonctive* set of numbers) of an actual number and is defined by an interval.

Other fuzzy cardinalities based on the definition of FGCounts, such as FLCounts or FECounts, have been defined by Zadeh or Wygralak [10,29,32]. Dubois, Prade [5] introduced a similar definition but they adopt a possibilistic point of view and a fuzzy cardinality is interpreted as a possibility distribution over α-cuts. The rest of this paper is based on the well known FGCounts because they completely satisfy the needs of our application domain.

In order to extend a binary operation # (e.g. $+$, $\times$, min, max, ...) [1] on $\mathbb{N}_\mathrm{f}$, we start from the operations on crisp bags based on arithmetic operations on integers. Then, due to the semantics associated with the degree of any integer ω in a fuzzy cardinality (the extent to which a fuzzy set has *at least* ω elements), these operations are extended to fuzzy integers by means of the generalized extension principle [29] defined by

$$\mu_{a\#b}(z) = \sup_{(x,y) \mid x\#y \geqslant z} \min(\mu_a(x), \mu_b(y)), \tag{5}$$

where a and b are two fuzzy natural integers.

Using α-cuts, a binary operation # on $\mathbb{N}_\mathrm{f}$ can be defined thanks to the corresponding operation on $\mathbb{N}$:

$$(a\#b)_\alpha = a_\alpha \# b_\alpha. \tag{6}$$

Example 2.1. We consider two fuzzy integers $a = \{1/0, 1/1, 1/2, 0.1/3\}$ and $b = \{1/0, 1/1, 0.5/2\}$. Using α-cuts, the minimum, addition and product operations can be easily performed. Hence, we obtain

$$\min(a, b) = \{1/0, 1/1, 0.5/2\},$$

$$a + b = \{1/0, 1/1, 1/2, 1/3, 0.5/4, 0.1/5\},$$

$$a \times b = \{1/0, 1/1, 1/2, 0.5/3, 0.5/4, 0.1/5, 0.1/6\}.$$

Due to the specific characterization of fuzzy integers (their characteristic function is monotonically decreasing on $[0, +\infty[$), it can easily be shown (using α-cuts) that $\mathbb{N}_\mathrm{f}$ is a semiring structure which means that operations of addition and product satisfied the following properties: $(\mathbb{N}_\mathrm{f}, +)$ is a commutative monoid ($+$ is closed and associative) with the neutral element $\{1/0\}$; $(\mathbb{N}_\mathrm{f}, \times)$ is a monoid with the neutral element $\{1/0, 1/1\}$; the product is distributive over the addition.

[1] In the rest of this paper we adopt an overloading principle and a binary operation on $\mathbb{N}_\mathrm{f}$, $\mathbb{Z}_\mathrm{f}$ or $\mathbb{Q}_\mathrm{f}$ is represented by the same symbol, such as $+$ for the addition.

2.3. Fuzzy bags operations based on $\mathbb{N}_f$

The concepts of degree and number of occurrences, which both characterize an element x in a fuzzy bag, can be simultaneously dealt through the concept of fuzzy integer. Considering this notion, the occurrences of an element x in a fuzzy bag A can be characterized as a fuzzy integer denoted by $\Omega_A(x)$. This fuzzy number is the fuzzy cardinality of the fuzzy set of the different occurrences of x in A. Thus, a fuzzy bag A, on a universe U, can be defined by a characteristic function Ω_A from U to $\mathbb{N}_f$:

$$\Omega_A : U \to \mathbb{N}_f.$$

Example 2.2. The characteristic of the element 'a' in the fuzzy bag $A = \{\langle 1, 0.1, 0.1\rangle/a,\ \langle 0.5\rangle/b\}$ is $\Omega_A(a) = \{1/0, 1/1, 0.1/2, 0.1/3\}$. Using fuzzy integers, A can alternatively be represented as $A = \{\{1/0, 1/1, 0.1/2, 0.1/3\}^*a, \{1/0, 0.5/1\}^*b\}$.

So, the α-cut of a fuzzy bag A can be defined as the crisp bag A_α such that the number of occurrences of the element x in A_α is an integer associated with the α-cut of the fuzzy number of occurrences of x in $A : \omega A_\alpha(x) = (\Omega_A(x))_\alpha$.

From the basic operations on fuzzy integers, operations on crisp bags can be straightforwardly extended to fuzzy bags [23,27]. Thus, the cardinality of a fuzzy bag A drawn from U, denoted by $|A|$, is defined by

$$|A| = \sum_{x \in U} \Omega_A(x) \tag{7}$$

and the extension of the operations over bags leads to

$$\Omega_{A\cap B}(x) = \min(\Omega_A(x), \Omega_B(x)), \tag{8}$$

$$\Omega_{A\cup B}(x) = \max(\Omega_A(x), \Omega_B(x)), \tag{9}$$

$$\Omega_{A+B}(x) = \Omega_A(x) + \Omega_B(x), \tag{10}$$

$$\Omega_{A\times B}(x) = \Omega_A(x) \times \Omega_B(x), \tag{11}$$

where binary operations (min, max, $+$, $\times$) on fuzzy integers are defined by (5), $A + B$ is the additive union and $A \times B$ is the cartesian product over two fuzzy bags A and B. Note that, due the particular shape of fuzzy integers, min or max can be performed "vertically" (by combining degrees) or horizontally (by combining integers).

In the special case where A and B are reduced to bags (resp. fuzzy sets), the fuzzy numbers $\Omega_A(x)$ and $\Omega_B(x)$ can be written as $\{0/1, 1/1, 1/2, \ldots, 1/n\}$ (resp. $\{1/0, \alpha/1\}$), formulas (7)–(11) yield to the usual specification of the corresponding operations over bags on $\mathbb{N}$ (resp. fuzzy sets on [0, 1]). Hence, $\mathbb{N}_f$ provides a general framework in which sets, bags, fuzzy sets and fuzzy bags can be represented through a common representation. Consequently, these structures can be composed thanks to a small number of generic operators.

Example 2.3. Let A and B be the two fuzzy bags represented as follows: $A=\{\{1/0, 1/1, 0.1/2, 0.1/3\}^*a, \{1/0, 0.5/1\}^*b\}$; $B = \{\{1/0, 0.9/1, 0.5/2\}^*a\}$.

The number of occurrences of the elements a and b in $A \cap B$ are characterized by

$$\Omega_{A\cap B}(a) = \min(\{1/0, 1/1, 0.1/2, 0.1/3\}, \{1/0, 0.9/1, 0.5/2\}) = \{1/0, 0.9/1, 0.1/2\},$$

$$\Omega_{A\cap B}(b) = \min(\{1/0, 0.5/1\}, \{1/0\}) = \{1/0\} = 0.$$

So, we deduce: $A \cap B = \{\{1/0, 0.9/1, 0.1/2\}^{*}a\}$ which is the same result as the one obtained using α-cuts.

The additive union between A and B leads to put together the elements of A and B and it produces the following fuzzy bag: $\{\langle 1, 0.9, 0.5, 0.1, 0.1\rangle/a, \langle 0.5\rangle b\}$ which can also be represented with fuzzy numbers of occurrences by $\{\{1/0, 1/1, 0.9/2, 0.5/3, 0.1/4, 0.1/5\}/a, \{1/0, 0.5/1\}/b\}$. This can be evaluated by adding the number of occurrences of a (resp. b) in A and the number of occurrences of a (resp. b) in B:

$$\begin{aligned}\Omega_{A+B}(a) &= \Omega_A(a) + \Omega_B(a) = \{1/0, 1/1, 0.1/2, 0.1/3\} + \{1/0, 0.9/1, 0.5/2\}\\ &= \{1/0, 1/1, 0.9/2, 0.5/3, 0.1/4, 0.1/5\}.\end{aligned}$$

$$\Omega_{A+B}(b) = \Omega_A(b) + \Omega_B(b) = \{1/0, 0.5/1\} + \{1/0\} = \{1/0, 0.5/1\}.$$

A fuzzy bag A is said to be a subbag of a fuzzy bag B if and only if there are at least ω occurrences of x in A then there are at least ω occurrences of x in B, for any $x \in X$ and $\omega \in \mathbb{N}^+$. Formally, we have

$$A \subseteq B \quad \text{iff} \quad \forall x \in X, \quad \forall \omega \in \mathbb{N}^+, \quad \mu_{\Omega_A(x)}(\omega) \leqslant \mu_{\Omega_B(x)}(\omega). \tag{12}$$

This yields to define a gradual inclusion in order to evaluate the extent to which a fuzzy bag A is a subbag of a fuzzy bag B. When the standard conjunction and Gödel implication are chosen, we get

$$\begin{aligned}\mu_{\subseteq}(A, B) = &\min_{x\in X}, \quad \min_{\omega\in\mathbb{N}^+} (\mu_{\Omega_A(x)}(\omega) \Rightarrow_f \mu_{\Omega_B(x)}(\omega)),\\ &\text{with} : (p \Rightarrow_f q) = 1 \text{ when } p \leqslant q,\\ &\qquad\qquad\qquad = q \text{ otherwise}.\end{aligned} \tag{13}$$

Example 2.4. Let A and B be two fuzzy bags: $A = \{\{1/0, 1/1, 0.9/2, 0.4/3\}^{*}a\}$; $B = \{\{1/0, 1/1, 0.8/2, 0.5/3\}^{*}a\}$. The extent to which A is included into B is evaluated by

$$\mu_{A\subseteq B} = \min(1 \Rightarrow_{\text{Gö}} 1, 1 \Rightarrow_{\text{Gö}} 1, 0.9 \Rightarrow_{\text{Gö}} 0.8, 0.4 \Rightarrow_{\text{Gö}} 0.5) = \min(1, 1, 0.8, 1) = 0.8.$$

Due to the Gödel implication semantics, the degree 0.8 is the threshold t such that $A_\alpha \subseteq B_\alpha, \forall \alpha \in]0, t]$.

Independently Miyamoto [21] proposed a characterization of fuzzy bags which is quite similar to our approach. Miyamoto propositions are also based on the property of distributivity of α-cuts which allows crisp bags to be considered as a special case of fuzzy bags (such a property is not always satisfied [27] by the original Yager's model [30]). In [21] an element x of a fuzzy bag A, is characterized by a decreasing sequence of membership degrees of the different occurrences of x in A. Thus, if x has p occurrences in A then x is characterized by the sequence of degrees: $(\mu_A^1(x), \mu_A^2(x), \ldots, \mu_A^p(x))$ with $\mu_A^1(x) \geqslant \mu_A^2(x) \geqslant \cdots \geqslant \mu_A^p(x)$. An operation (for example $\cap$) between two fuzzy bags A and B

is defined by combining degrees of the same rank $j(\mu^j_{A\cap B}(x) = \mu^j_A(x) \wedge \mu^j_B(x))$. It is clear that a membership sequence $(\mu^1_A(x), \mu^2_A(x), \ldots, \mu^p_A(x))$ can be viewed as a representation of a fuzzy number of occurrences $\Omega_A(x) = \{\mu^1_A(x)/1, \mu^2_A(x)/2, \ldots, \mu^p_A(x)/p\}$ and that $\Omega_{A\cap B}(x) = \min(\Omega_A(x), \Omega_B(x))$ leads to combinations $\mu^j_{A\cap B}(x) = \mu^j_A(x) \wedge \mu^j_B(x))$ because the characteristic functions of fuzzy integers $\Omega_A(x)$ and $\Omega_B(x)$ are decreasing. However, it seems very important to us to put forward the concept of fuzzy integer (not only sequences of degrees) which both generalizes the notions of integer and degree with their associated operators. Thus, operations over collections (sets, fuzzy sets, bags or fuzzy bags) are treated in a similar way (because they can be defined through a common mechanism: fuzzy cardinalities) and, consequently, the algebra over these structures is still reduced to a small number of operators. Moreover, the emergence of the concept of fuzzy integer can be enlarged to other structures such as $\mathbb{Z}_\mathrm{f}$ or $\mathbb{Q}_\mathrm{f}$ (it is our objective in this paper) which provide foundations for dealing with problems about absolute or relative quantifications. Finally, in [21], Miyamoto defines the cardinality of fuzzy bag A on a universe U, by $\sum_{x\in U}\sum_j \mu^j_A(x)$. This definition is derived from a ΣCount approach which can be viewed as an approximation of a fuzzy cardinality. In our approach, the cardinality of A is naturally defined by a *fuzzy cardinality*: $|A| = \sum_{x\in U} \Omega_A(x)$ providing an *exact* (but *gradual*) representation of the cardinality of A. From this cardinality it is then possible to evaluate an *approximation* on $\mathbb{R}$ thanks to a Lebesgue integral (similarly to the method used in Sections 4.3.A and 5.3 in this paper).

3. Fuzzy relative integers

In Section 2 we have shown that $(\mathbb{N}_\mathrm{f}, +)$ is a monoid, in this section, we extend $\mathbb{N}_\mathrm{f}$ to $\mathbb{Z}_\mathrm{f}$ in order to build up a group structure.

Let us consider the equivalence relation $\mathcal{R}$ such that

$$\forall (x^+, x^-) \in \mathbb{N}_\mathrm{f} \times \mathbb{N}_\mathrm{f}, \quad \forall (y^+, y^-) \in \mathbb{N}_\mathrm{f} \times \mathbb{N}_\mathrm{f}, \quad (x^+, x^-)\mathcal{R}(y^+, y^-) \quad \text{iff} \quad x^+ + y^- = x^- + y^+, \tag{14}$$

where $+$ is the addition on $\mathbb{N}_\mathrm{f}$. The set of fuzzy relative integers is defined by

$$\mathbb{Z}_\mathrm{f} = (\mathbb{N}_\mathrm{f} \times \mathbb{N}_\mathrm{f})/\mathcal{R}, \tag{15}$$

which is the quotient set of all equivalence classes on $(\mathbb{N}_\mathrm{f} \times \mathbb{N}_\mathrm{f})$ defined by $\mathcal{R}$.

Example 3.1. Let a and b be two fuzzy naturals integers: $a = \{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}$; $b = \{1/0, 1/1, 0.3/2\}$. Then the following pair (a, b) is one instance of an equivalence class which defines a fuzzy relative integer:

$$(a, b) = (\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}, \{1/0, 1/1, 0.3/2\}).$$

Other instances of this class could be

$$(a', b') = (\{1/0, 0.8/1, 0.5/2, 0.2/3\}, \{1/0, 0.3/2\}),$$

$$(a'', b'') = (\{1/0, 1/1, 0.9/2, 0.8/3, 0.5/3, 0.2/4\}, \{1/0, 1/1, 0.9/2, 0.3/3\}).$$

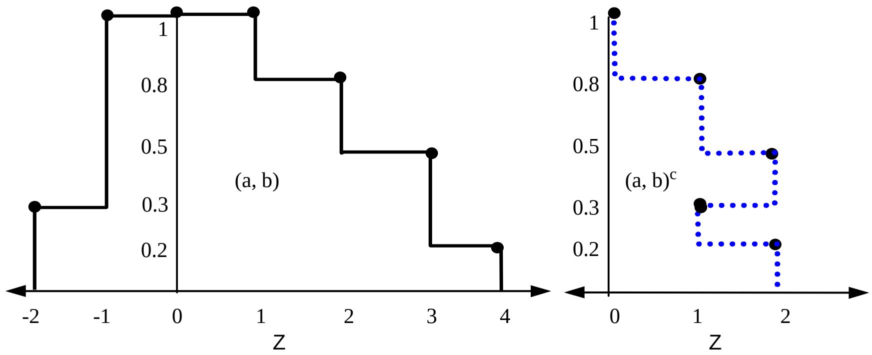

Fig. 2. A graphical representation of a fuzzy relative integer (a, b) and its compact representation $(a, b)^c$.

Each α-cut of a fuzzy relative integer (x^+, x^-) is a pair of positive integers (x^+_α, x^-_α) which can be interpreted as a relative integer $(x^+_\alpha - x^-_\alpha)$. Consequently, any fuzzy relative integer x has a unique canonical representative x^c which can be obtained by enumerating the values of its different α-cuts on $\mathbb{Z}$:

$$x^c = \sum \alpha_i/(x^+_{\alpha_i} - x^-_{\alpha_i}), \tag{16}$$

where α_i's correspond to the different α-cuts of x.

Example 3.2. The compact denotation of the fuzzy relative (a, b) (cf. Example 3.1) is $(a, b)^c = \{1/0, 0.8/1, 0.5/2, 0.3/1, 0.2/2\}^c$ which is graphically represented in Fig. 2.

Let $(x, y) \in \mathbb{Z}_f \times \mathbb{Z}_f$, the addition $(+)$ and the product $(\times)$ on $\mathbb{Z}_f$ are defined by

$$x + y = (x^+, x^-) + (y^+, y^-) = (x^+ + y^+, x^- + y^-), \tag{17}$$

$$x \times y = (x^+, x^-) \times (y^+, y^-) = ((x^+ \times y^+) + (x^- \times y^-), (x^+ \times y^-) + (x^- \times y^+)). \tag{18}$$

As the arithmetic operations on $\mathbb{N}_f$ can be defined in terms of operations on their α-cuts, an operation # (+ or $\times$) on $\mathbb{Z}_f$ is also compatible with α-cuts and the following property holds:

$$(x\#y)_\alpha = x_\alpha \# y_\alpha, \tag{19}$$

where the operation # (in the right part of this expression) is an operation on $\mathbb{Z}$.

The addition is commutative, associative and has a neutral element, denoted by $0_{\mathbb{Z}_f}$, defined by the class $\{(a, a)/a \in \mathbb{N}_f\}$.

Each fuzzy relative integer $x = (x^+, x^-)$ has an opposite, denoted by $-x = (x^-, x^+)$, such that $x + (-x) = (x^+ + x^-, x^- + x^+)$ which is exactly $0_{\mathbb{Z}_f}$ (+ is commutative). This property is remarkable in comparison with the framework of usual fuzzy numbers where 'approximately 1' minus 'approximately 1' returns a value corresponding to 'approximately 0' which is not exactly 0. Consequently, the difference operation on $\mathbb{Z}_f$ is given by

$$x - y = x + (-y) = (x^+ + y^-, x^- + y^+). \tag{20}$$

So, in $\mathbb{Z}_f$, the difference between two numbers is always defined and can be represented by a unique canonical representative [25].

It can be easily checked that this product is commutative, associative and distributive over the addition. The neutral element, denoted by $1_{\mathbb{Z}_f}$, is the fuzzy relative integer $(\{1/0, 1/1\}, \{1/0\})$. Therefore we conclude that $(\mathbb{Z}_f, +, \times)$ forms a ring.

In bag theory, the difference between two bags A and B is defined as the relative complement of $A \cap B$ with respect to A. Unfortunately, on $\mathbb{N}_f$, the difference operation between two bags A and B cannot always be computed because an element x of $A - B$ is not always characterized by a positive fuzzy integer. Thus, in the continuity of propositions of Blizard [2] and Chakrabarty [6,7], where the concept of shadow bag is defined as a generalization of the concept of bag, in [25] we have shown that $\mathbb{Z}_f$ provides a sound framework in which the generalization of fuzzy bags is well-founded and where the difference of two fuzzy bags can always be defined.

4. Fuzzy rational numbers

On $\mathbb{Z}_f$, only the elements of classes $(1, 0)$ and $(0, 1)$ have a multiplicative reciprocal. We now extend $\mathbb{Z}_f$ to $\mathbb{Q}_f$ in order to achieve field properties and to create a commutative ring in which every nonzero element is invertible.

4.1. Definition of $\mathbb{Q}_f$

Let $\mathbb{Z}_f^{**}$ be the set of fuzzy relative integers such that: $\forall z \in \mathbb{Z}_f^{**}, \forall \alpha \in]0, 1], z_\alpha \neq 0$. Mathematically, we may define a fuzzy rational number as a pair of fuzzy relative integers $[x^n, x^d] \in \mathbb{Z}_f \times \mathbb{Z}_f^{**}$ and an equivalence relation $\mathcal{R}'$ upon such pairs specified by the rule:

$$\forall (x^n, x^d) \quad \text{and} \quad (y^n, y^d) \in \mathbb{Z}_f \times \mathbb{Z}_f^{**}, \quad [x^n, x^d]\mathcal{R}'[y^n, y^d] \quad \text{iff} \quad x^n \times y^d = y^n \times x^d. \tag{21}$$

The set of fuzzy rational numbers is then

$$\mathbb{Q}_f = (\mathbb{Z}_f \times \mathbb{Z}_f^{**})/\mathcal{R}', \tag{22}$$

it is the quotient set of all equivalence classes on $(\mathbb{Z}_f \times \mathbb{Z}_f^{**})$ defined by $\mathcal{R}'$.

An instance of a fuzzy rational number denoted by $[x^n, x^d]$ can also be rewritten with fuzzy natural integers: $[(x^{n+}, x^{n-}), (x^{d+}, x^{d-})]$.

The α-cut of a fuzzy rational number x is defined by

$$\forall \alpha \in]0, 1], \quad x_\alpha = [x^n_\alpha, x^d_\alpha] = [(x^{n+}_\alpha, x^{n-}_\alpha), (x^{d+}_\alpha, x^{d-}_\alpha)]. \tag{23}$$

Because of the distributivity of the α-cut function over the addition and the multiplication on $\mathbb{Z}_f$, if x and y are two fuzzy rational numbers, their α-cuts are compatible with the equivalence relation $\mathcal{R}'$, and then

$$[x^n, x^d]\mathcal{R}'[y^n, y^d] \quad \text{iff} \quad \forall \alpha \in]0, 1], \quad x^n_\alpha \times y^d_\alpha = x^d_\alpha \times y^n_\alpha, \quad \forall \alpha \in [0, 1]. \tag{24}$$

The representation of a fuzzy relational number x with a couple of fuzzy natural integers is not very tractable. So, let us now introduce a more convenient notation. We use a more simple compact representation (denoted by x^c) by enumerating values associated with the different α-cuts which are rationals. With such a representation a fuzzy relational number can be defined by

$$x^{\mathrm{c}} = \sum \alpha/(x_\alpha^{n+} - x_\alpha^{n-}) \div (x_\alpha^{d+} - x_\alpha^{d-}). \tag{25}$$

If all ratios $(x_\alpha^{n+} - x_\alpha^{n-}) \div (x_\alpha^{d+} - x_\alpha^{d-})$ are reduced, then we get a canonical compact form of x.

Example 4.1. Let us consider the two fuzzy positive relative integers:

$$(x^{n+}, x^{n-}) = (\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}, 0); (x^{d+}, x^{d-}) = (\{1/0, 1/1, 0.3/2\}, 0).$$

The pair $[(x^{n+}, x^{n-}), (x^{d+}, x^{d-})]$ represents a fuzzy rational number x which belongs to $\mathbb{Q}_{\mathrm{f}}$. Such a number can be written in many other forms, for example

$$[(y^{n+}, y^{n-}), (y^{d+}, y^{d-})] = [(\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4, 0.2/5, 0.2/6\}, 0), (\{1/0, 1/1, 0.3/2, 0.2/3\}, 0)].$$

As a matter of fact, x is equivalent to y by $\mathcal{R}'$ because

$$x^{n+} \times y^{d+} = x^{d+} \times y^{n+} = \{1/0, 1/1, 0.8/2, 0.5/3, 0.3/4, 0.3/5, 0.3/6, 0.2/7, 0.2/8, 0.2/9, 0.2/10, 0.2/11, 0.2/12\}.$$

For all α-cuts in $\{1, 0.8, 0.5, 0.3, 0.2\}$, we get $((x_\alpha^{n+} \times y_\alpha^{d+}), 0) = ((x_\alpha^{d+} \times y_\alpha^{n+}), 0)$. This means that $x_\alpha^{n+} \div x_\alpha^{d+} = y_\alpha^{n+} \div y_\alpha^{d+}$. These ratios are respectively: $1 \div 1$, $2 \div 1$, $3 \div 1$, $3 \div 2$ and $4 \div 2$. Thus, x and y have the same canonical compact representation defined by

$$\{1/1, 0.8/2, 0.5/3, 0.3/3 \div 2, 0.2/2\}^{\mathrm{c}}.$$

4.2. Operations on $\mathbb{Q}_{\mathrm{f}}$

The addition and product of two fuzzy rational numbers x and y, represented by pairs of relative integers, $[x^n, x^d]$ and $[y^n, y^d]$, are defined by the following rules:

$$[x^n, x^d] + [y^n, y^d] = [(x^n \times y^d) + (y^n \times x^d), x^d \times y^d], \tag{26}$$

$$[x^n, x^d] \times [y^n, y^d] = [x^n \times y^n, x^d \times y^d]. \tag{27}$$

These definitions extend the usual crisp definitions and are compatible with the concept of α-cuts. The addition is commutative, associative and has an additive identity $[0_{\mathbb{Z}_{\mathrm{f}}}, 1_{\mathbb{Z}_{\mathrm{f}}}]$, denoted by $0_{\mathbb{Q}_{\mathrm{f}}}$, such that

$$[x^n, x^d] + [0_{\mathbb{Z}_{\mathrm{f}}}, 1_{\mathbb{Z}_{\mathrm{f}}}] = [(x^n \times 1_{\mathbb{Z}_{\mathrm{f}}}) + (0_{\mathbb{Z}_{\mathrm{f}}} \times x^d), x^d \times 1_{\mathbb{Z}_{\mathrm{f}}}] = [x^n, x^d].$$

The product is commutative, associative and has an neutral element $[1_{\mathbb{Z}_{\mathrm{f}}}, 1_{\mathbb{Z}_{\mathrm{f}}}]$, denoted by $1_{\mathbb{Q}_{\mathrm{f}}}$, corresponding to the class: $\{[a, a]/a \in \mathbb{Z}_{\mathrm{f}}^{**}\}$:

$$[x^n, x^d] \times [a, a] = [x^n \times a, x^d \times a] \equiv [x^n, x^d].$$

For each element $x = [x^n, x^d]$ belonging to $\mathbb{Q}_{\mathrm{f}}^{**} = (\mathbb{Z}_{\mathrm{f}}^{**} \times \mathbb{Z}_{\mathrm{f}}^{**}))/\mathcal{R}'$ (which means that $\forall x \in \mathbb{Q}_{\mathrm{f}}^{**}, \forall \alpha \in]0, 1], x_\alpha \neq 0$), there exists an inverse $x^{-1} = [x^d, x^n]$, such that $x \times x^{-1} = 1_{\mathbb{Q}_{\mathrm{f}}}$:

$$x \times x^{-1} = [x^n \times x^d, x^d \times x^n] \equiv [1_{\mathbb{Z}_{\mathrm{f}}}, 1_{\mathbb{Z}_{\mathrm{f}}}] = 1_{\mathbb{Q}_{\mathrm{f}}} \quad (\times \text{ is commutative}).$$

Table 1
Division and α-cuts

α	Dividend_α	Divisor_α	$(\text{Dividend} \div \text{Divisor})_\alpha$
1	1	1	1
0.8	2	1	2
0.5	3	1	3
0.3	3	2	$3 \div 2$
0.2	4	2	2

Consequently we can define an exact division operation, $\div$, on $\mathbb{Q}_\mathrm{f} \times \mathbb{Q}_\mathrm{f}^{**}$, such that

$$x \div y = x \times y^{-1} = [x^n \times y^d, x^d \times y^n]. \tag{28}$$

With such a definition the following property holds on $\mathbb{Q}_\mathrm{f}$:

$$y \times (x \div y) = x.$$

This property is remarkable because it is not always satisfied if we consider "ordinary fuzzy numbers" representing ill-known quantities based on an extension of the interval framework.

Proof. The division between two fuzzy rational numbers x and y is defined by $x \div y = [x^n \times y^d, x^d \times y^n]$. So, the product $y \times (x \div y)$ is $[y^n \times x^n \times y^d, y^d \times x^d \times y^n]$. Such a result is equivalent to $[x^n, x^d]$ by $\mathcal{R}'$ because: $y^n \times x^n \times y^d \times x^d = y^d \times x^d \times y^n \times x^n$ ($\times$ is commutative).

Example 4.2. Let us consider the two fuzzy relative integers:

$$\text{dividend} = (\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}, 0) = \{1/1, 0.8/2, 0.4/3, 0/2/4\}^\mathrm{c},$$

$$\text{divisor} = (\{1/0, 1/1, 0.3/2\}, 0) = \{1/1, 0.3/2\}^\mathrm{c}.$$

The division (dividend $\div$ divisor) can be evaluated by α-cuts. From Table 1 we deduce the compact form of the exact division (dividend $\div$ divisor) on $\mathbb{Q}_\mathrm{f}$: $\{1/1, 0.8/2, 0.5/3, 0.3/3 \div 2, 0.2/2\}^\mathrm{c}$. Fig. 3 is a graphical representation of this result.

Note that, if dividend equals divisor the result is $\{1/0, 1/1\}$.

4.3. *Approximations of the exact division on $\mathbb{Q}_\mathrm{f}$*

The main interest of $\mathbb{Q}_\mathrm{f}$ is to provide an algebraic basis allowing exact divisions and calculus compositions. From an exact result evaluated on $\mathbb{Q}_\mathrm{f}$, it is then possible to perform different approximations, for examples on $\mathbb{N}_\mathrm{f}$ or $\mathbb{R}$, depending on application domain needs. This subsection shows how such these approximations (or summarizations) can be extracted from an exact calculation.

4.3.1. *Euclidian division*

On $\mathbb{N}_\mathrm{f}$, the difference between two fuzzy integers a and b is not always defined, even if a is greater then b. For example, let us consider $a = \{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}$ and $b = \{1/0, 1/1, 0.3/2\}$, there is

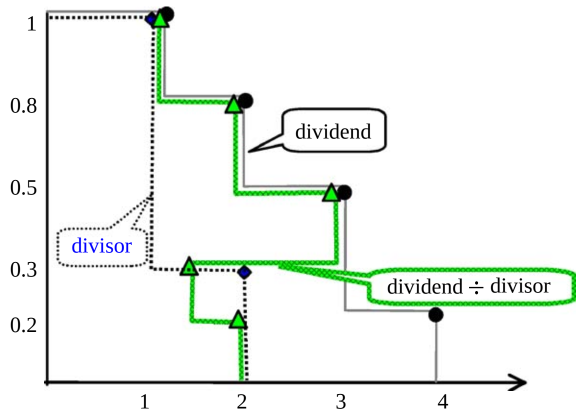

Fig. 3. A graphical representation of the exact division (dividend ÷ divisor).

no s such that: $a = b + s$. This leads us to define an approximate difference [24], noted a)-(b, as the greatest fuzzy integer s satisfying $b + s \leqslant a$.

Such a definition specifies the best lower approximation of the difference between two fuzzy natural integers. In the domain of fuzzy equations this operation)-(is called an optimistic difference [28]. We have noticed that the difference denoted by)-(allows to specify the best lower approximation on $\mathbb{N}_f$ of the exact difference on $\mathbb{Z}_f$.

In the same way, a division denoted by)÷(, similar to the optimistic division defined by Sanchez, allows to specify the best lower approximation on $\mathbb{N}_f$ of the exact division on $\mathbb{Q}_f$. This division is such that, for two fuzzy numbers dividend and divisor, $Q =$ dividend)÷(divisor is the *greatest* fuzzy natural integer q such that divisor $\times q \leqslant$ dividend. This is an extension of the Euclidian division.

Such a result can be iteratively evaluated by using the difference on $\mathbb{N}_f$ as illustrated in the following algorithm:

$R :=$ dividend;
$Q := 0$;
while (divisor $\leqslant R$) **do begin**
  $R := R$)-(divisor;
  $Q := Q + 1$
endo.

At this step the value of Q is the greatest positive integer q such that

$$(\text{divisor} \times q + R = \text{dividend}) \text{ and } \textit{not } (\text{divisor} \leqslant R).$$

The value of Q is the *integer multiplicity* factor between dividend and divisor. We can go further by evaluating a *fuzzy* multiplicity factor.

As fuzzy integers form a lattice, the property: *not* (divisor $\leqslant R$) does not imply (divisor $> R$). For example, when divisor $= \{1/0, 1/1, 1/2, 0.3/3\}$ and $R = \{1/0, 1/1, 0.8/2, 0.5/3\}$. Let α_1 be the extent to which divisor $\leqslant R$ (i.e.: α_1 is the greatest threshold such that: $\forall \alpha \in]0, \alpha_1]$, divisor$_\alpha \leqslant R_\alpha$ [23] or, in other words, α_1 is the greatest degree such that: $\{1/0, \alpha_1/1\} \times$ divisor $\leqslant R$). The difference $R := R - (\{1/0, \alpha_1/1\} \times \text{divisor})$ can be performed and repeated until the predicate ($R <$ divisor) becomes true. This leads to

$$\begin{aligned}\text{dividend} &= q \times \text{divisor} + (\{1/0, \alpha_1/1\} \times \text{divisor}) + \cdots + (\{1/0, \alpha_n/1\} \times \text{divisor}) + R \\ &= \{1/0, 1/1, \ldots, 1/q, \alpha_1/q+1, \ldots, \alpha_n/q+n\} \times \text{divisor} + R \text{ and } (\text{divisor} > R). \quad (29)\end{aligned}$$

Consequently, the fuzzy number quotient $Q = \{1/0, 1/1, \ldots, 1/q, \alpha_1/q+1, \ldots, \alpha_n/q+n\}$ is the *greatest* positive fuzzy integer q such as divisor $\times\, q \leqslant$ dividend.

This decomposition helps us to understand the meaning of a fuzzy multiplicity factor on $\mathbb{N}_f$ defined by a fuzzy integer quotient such as $\{1/0, \alpha_1/1, \ldots, \alpha_n/n\}$. In such a number a degree α_i is the extent to which ($i \times$ *divisor*) *is smaller* than dividend.

Example 4.3. Let us consider the values of dividend and divisor defined in Example 4.2:

$$\text{dividend} = \{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\};\ \text{divisor} = \{1/0, 1/1, 0.3/2\}.$$

The Euclidian division using successive differences is given by computing R_1 and R_2:

$$R_1 = \text{dividend })\text{-}(\text{ divisor} = \{1/0, 0.8/1, 0.2/2\},$$

as the extent to which divisor is smaller than R_1 is $\min(1 \Rightarrow_f 1, 1 \Rightarrow_f 0.8, 0.3 \Rightarrow_f 0.2) = 0.2$, it is possible to subtract the divisor from R_1 but just ‘until’ the α-cut 0.2:

$$R_2 = R_1)\text{-}([\{1/0, 0.2/1\} \times \text{divisor}] = \{1/0\}.$$

We then deduce

$$\text{dividend })\div(\text{ divisor} = \{1/0, 1/1\} + \{1/0, 0.2/1\} = \{1/0, 1/1, 0.2/2\}.$$

The product: $\{1/0, 1/1, 0.3/2\} \times \{1/0, 1/1, 0.2/2\} = \{1/0, 1/1, 0.2/2, 0.2/3, 0.2/4\}$ is smaller than $\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}$ and $\{1/0, 1/1, 0.2/2\}$ is the greater fuzzy natural number satisfying such a property. This result is a fuzzy multiplicity factor where the degree 0.2 means the extent to which $2 \times$ divisor is smaller than dividend.

The Euclidian division (dividend)÷(divisor) is the best approximation by lower value on $\mathbb{N}_f$ of the exact division (dividend ÷ divisor) on $\mathbb{Q}_f$ represented by $\{1/1, 0.8/2, 0.5/3, 0.3/3 \div 2, 0.2/2\}^c$ (see Fig. 4).

4.3.2. Other approximations

In the previous paragraph, the Euclidian division has been defined as a lower approximation on $\mathbb{N}_f$ of the exact division (dividend ÷ divisor) on $\mathbb{Q}_f$. In a symmetric way, as shown in Fig. 4, an another estimation could be given by computing the best upper value on $\mathbb{N}_f$ of the exact division. These two approximations provide an interval on $\mathbb{N}_f$ which estimates the exact division.

A scalar counterpart of an exact division on $\mathbb{Q}_f$ can be given by the expected value [13,14] given by a Lebesgue integral:

$$E(\text{dividend} \div \text{divisor}) = \sum_{i=1}^{n} (\text{dividend} \div \text{divisor})_{\alpha_i} \times (\alpha_i - \alpha_{i+1}), \quad (30)$$

where the degrees α_i are decreasingly ordered: $1 = \alpha_1 > \alpha_2 > \cdots > \alpha_{n-1} > \alpha_n = 0$.

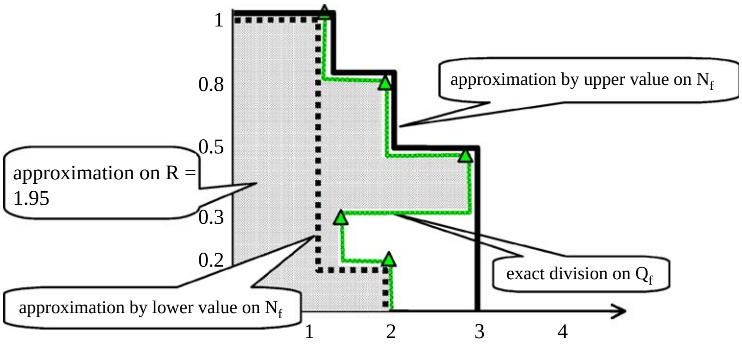

Fig. 4. Different approximations of an exact division on $\mathbb{Q}_f$.

Such a measure is an approximation on $\mathbb{R}$ of the exact division. It gives an idea of the real shape of the exact division.

Example 4.4. The expected value of the division (dividend)÷(divisor) (cf. Example 4.3) on $\mathbb{R}$ is

$$(1 - 0.8) \times 1 + (0.8 - 0.5) \times 2 + (0.5 - 0.3) \times 3 + (0.3 - 0.2) \times 1.5 + 0.2 \times 2 = 1.95$$

and the different approximations of the exact result can be graphically represented by Fig. 4.

5. Division operations and some applications to flexible querying

This work takes place in the study of a query language which allows the formulation of imprecise queries addressed to databases. In this framework, this section first proposes generalizations of the relational division thanks to an Euclidian division. Next, the interest of the arithmetic on $\mathbb{Q}_f$ is illustrated through the management of a query involving an average function or the evaluation of comparison indices of fuzzy collections.

5.1. Generalization of the relational division

The notion of division is well known in the relational model of data. For example, the query looking for *the stores having in stock all the ordered products* is based on a division which finds the stores such that the set of their stored products contains the set of ordered products.

A first extension may be imagined by introducing some graduality in the previous query, which would be: *find the stores having among their recent stored products all the expensive ordered products*. This kind of query has been studied in the relational framework [4] using fuzzy relations.

Other data models, for example object oriented models, support complex data structures, such as bags. For example, collections of ordered products or stored products can account for numbers of occurrences of their elements and can be modeled by bags. In this situation, a query corresponding to a division would then be: *find the stores having in stock all the ordered products and the number of times each of them can deliver this order.*

More generally, a model supporting fuzzy bags takes advantage of both graduality and number of occurrences. For example, the stock of a store can be modeled as a fuzzy bag where each copy of a product is associated with a degree of membership because it may have been stored in more or less recently. In this case, the assessment of the number of times (a bag of) ordered products are contained in (a fuzzy bag of) recently stored products leads to define a fuzzy factor of multiplicity.

In the following, we first analyze the division on sets and fuzzy sets in the framework of the relational model, then we show how this operator can be generalized on bags and fuzzy bags.

5.1.1. Divisions on sets and fuzzy sets

In the context of the relational model of data, a universe is modeled as a set of relations R_i which are subsets of the Cartesian product of some domains. The relational division of $R(A, X)$ by $S(A, Y)$, denoted by $R[A \div A]S$, where A is a set of attributes common to R and S, aims at determining the X-values connected in R with all the A-values appearing in S. Formally this operation can be defined by

$$x \in R[A \div A]S \Leftrightarrow \forall s, \quad s \in S[A] \Rightarrow (x, s) \in R. \tag{31}$$

If $S' = \{s | s \in S[A]\}$ and $x.R' = \{s | (x, s) \in R\}$ then $R[A \div A]S$ can also be defined by

$$x \in R[A \div A]S \Leftrightarrow S' \subseteq x.R'. \tag{32}$$

For example, if we know the stores which keep in stock products (relation R), on the one hand, and ordered products (relation S), on the other hand, the query looking for *the stores able to deliver all the ordered products* is a division of R by S. These stores are such that the set of the ordered products is *included* in the set of their stored products.

A generalization of the relational division, based on fuzzy relations R and S, has been proposed in [4]. The expression (31) has been extended by changing the usual implication into a multiple-valued one and interpreting the universal quantifier as a generalized conjunction:

$$\mu_{R[A \div A]S}(x) = \inf_S (\mu_S(s) \Rightarrow_f \mu_R(x, s[A])). \tag{33}$$

As the implication models an inclusion, the division can be characterized by a degree of inclusion:

$$\mu_{R[A \div A]S}(x) = \mu_{\subseteq_f}(S', x.R'). \tag{34}$$

With the Gödel's inclusion (based on the Gödel's implication), $\mu_{\subseteq_f}(S', x.R')$ is the greatest threshold such that: $\forall \alpha \in]0, t]$, $S'_\alpha \subseteq x.R'_\alpha$.

For example, S can be a fuzzy relation which represents the expensive ordered products and R can depict stores and their products recently kept in stock. A query involving a division on these two fuzzy relations could be: *find the stores having in their stock of recently stocked products all the ordered expensive products.* So, we are looking for stores x such that the fuzzy set of expensive ordered products (S') is *included* in the fuzzy set of their recent products ($x.R'$). In the result, the degree of membership of a store x expresses the extent to which S' is included in $x.R'$.

5.1.2. Divisions on bags and fuzzy bags

We now assume that our data model supports two collection types which model bags and fuzzy bags. It could be, for example, an object oriented model. Thus, we consider that the previous collections S' and $x.R'$ can be either bags or fuzzy bags. The question is: how to take advantage of their ability to manage quantities when we deal with a query involving a division? Following this aim, we propose some generalizations of the relational division.

5.1.2.1. Division operators on bags. We first consider that S' and $x.R'$ are bags. For example, S' can represent a bag of ordered products. S' is a bag because each product s is now associated with a number of occurrences, denoted by $\omega_{S'}(s)$, corresponding to the number of copies which have been ordered. The bag $x.R'$ is a bag of products kept in stock in a store x. The number of copies of a product s in this stock is denoted by $\omega_{x.R'}(s)$. We are concerned with queries such as: *find the stores having enough stored products to deliver all the ordered products*.

So, we are looking for every store x satisfying the following constraint:

$$\forall s \in S', \quad \omega_{S'}(s) \leqslant \omega_{x.R'}(s), \tag{35}$$

which means that the number of occurrences of a stored product in x is greater than the quantity of the corresponding product in the order. These stores are such that the bag of the ordered products is *included* in the bag of their stored products. The expression corresponds to the usual inclusion between bags.

However, the model allows a richer expressiveness, it can take into account the multiplicity factor between the number of occurrences of an ordered product and the number of copies of this product in a store x. This multiplicity is defined by

$$\omega_{x.R'}(s) \text{ div } \omega_{S'}(s), \tag{36}$$

where div is the (usual) Euclidian division. This operator allows to deal with queries such as: *find the number of times a store x can deliver all the ordered products*.

From that it follows a definition of the division of bags expressing *the number of times* a bag *is included* in another bag. This number is a multiplicity factor of the inclusion between two bags (denoted $\omega_{\subseteq}$) and is defined by

$$\omega_{\subseteq}(S', x.R') = \inf_{S'} (\omega_{x.R'}(s) \text{ div } \omega_{S'}(s)). \tag{37}$$

This formula derives from (33) where the usual implication is changed into an Euclidian division and the universal quantifier is interpreted as a generalized conjunction, here defined as the *infimum* on integers.

5.1.2.2. Division operators on fuzzy bags. We study the most general case and we now suppose that S' and $x.R'$ are two fuzzy bags. For example, the fuzzy bag $x.R'$ is a collection of more or less recent products kept in stock by x. Because each copy of a product has been stored in a more or less recent past and is associated with a level of freshness, each product is characterized by a fuzzy number of occurrences in $x.R'$ denoted by $\Omega_{x.R'}(s)$. The fuzzy bag S' may be a fuzzy bag of ordered products. Because it is possible to order different more or less fresh occurrences of a product s, each product in S' is associated with a fuzzy number of occurrences denoted by $\Omega_{S'}(s)$. For example, the fuzzy number $\Omega_{S'}(s) = \{1/0, 1/1, 0.3/2\}$ expresses that the user orders two occurrences of s, the expected freshness of the first one must be perfect, and the quality level of the second one has to be atleast higher than (or equal) 0.3.

Once again, we are concerned with queries such as: *find the number of times a store x can deliver, from its stock of more or less fresh copies of products, all the ordered copies of products, taking into account* *the minimum quality level of the expected copies*. So, we now consider both numbers of occurrences and degrees (relative to the freshness) which is interpreted as a threshold to be attained by a corresponding occurrence of a stored product.

The extension of expression (37) leads to the following formula based on an (fuzzy) Euclidian division applied on fuzzy numbers:

$$\Omega_{\subseteq}(S', x.R') = \inf_{S'}(\Omega_{x.R'}(s)) \div (\Omega_{S'}(s)). \tag{38}$$

The fuzzy number $\Omega_{\subseteq}(S', x.R')$ defines the fuzzy multiplicity factor between two fuzzy bags. It is *the greatest fuzzy number n* such that, when multiplying all the numbers of occurrences of the elements of S' by n, a fuzzy bag *included* in $x.R'$ is obtained.

Example 5.1. If the stock of the store x is characterized by the following fuzzy bag $x.R' = \{\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}^{*}s\}$ and the user's order is characterized by $S' = \{\{1/0, 1/1, 0.3/2\}^{*}s\}$, the number of times this order can be provided by x is

$$\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}) \div (\{1/0, 1/1, 0.3/2\} = \{1/0, 1/1, 0.2/2\}.$$

Such a fuzzy multiplicity factor means that the store x can deliver one time two occurrences of s with the expected levels of quality. Two other occurrences can be delivered but, in this case, the extent to which the user is satisfied is 0.2.

As matter of fact, if the order is delivered one time, the rest of stored products becomes $\{1/0, 1/1, 0.8/2, 0.5/3, 0.2/4\}) - (\{1/0, 1/1, 0.3/2\} = \{1/0, 0.8/1, 0.2/2\}$. As the extent to which $\{1/0, 1/1, 0.3/2\}$ is included in $\{1/0, 0.8/1, 0.2/2\}$ is given by $\min(1 \Rightarrow_f 0.8, 0.3 \Rightarrow_f 0.2) = 0.2$ (using Gödel implication), it can be deduced that the order can be delivered once again but only with the user satisfaction 0.2.

So, the inclusion of S' in $x.R'$ is characterized by the fuzzy multiplicity factor on $\mathbb{N}_f = \{1/0, 1/1, 0.2/2\}$ where the degree 0.2 expresses the extent to which S' is included in $x.R'$ two times.

To conclude this subsection, we notice that expression (38) is the most general expression of the division between collections $x.R'$ and S'. As a matter of fact, a degree of membership of an element x in a fuzzy set $A(\mu_A(x))$ is a special case of a fuzzy number (denoted: $\{0/1, 1/\mu_A(x)\}$), and a number of occurrences of an element x in a crisp bag A $(\omega_A(x))$ is a special case of a fuzzy number (denoted: $\{1/0, \ldots, 1/\omega_A(x)\}$). When (38) is applied to these specific fuzzy numbers (respectively degrees and integers) we can easily check that the obtained results are consistent with (33) and (38).

5.2. Division and fuzzy average

In databases, we may have to evaluate *the average of the salaries of young people* where salaries are precisely known. In this subsection, we illustrate that the evaluation of an average function, as proposed by Dubois and Prade [14], can also be viewed as an evaluation of a weighted mean on $\mathbb{Q}_f$.

Dubois and Prade propose to extend a set-function, such as the average function, to a fuzzy set F as a random number defined by

$$\text{average}(F) = \{\text{average}(F_{\alpha_i}), m(F_{\alpha_i}))/i = 1, 2, \ldots, n\},$$

where α_i's are the non-zero membership degrees in F decreasingly ordered ($\alpha_1 = 1 > \alpha_2, \ldots > \alpha_n > \alpha_{n+1} = 0$) and $m(F_{\alpha_i}) = \alpha_i - \alpha_{i+1}$.

The scalar counterpart of this definition is the expected value given by

$$E(\text{average}(F)) = \sum_{i=1}^{n} (\alpha_i - \alpha_{i+1}) \times \text{average}(F_{\alpha_i}). \tag{39}$$

Example 5.2. Let us consider the following fuzzy set of persons $Y = \{1/p1, 1/p2, 0.4/p3, 0.1/p4\}$. We suppose that their salaries are respectively: 2000, 1000, 2000 and 1000 euros. The average of salaries of people in F_{α_i} for the different a-cuts are:

α	F_{α_i}	Average (F_{α_i})
1	{2000, 1000}	1500
0.4	{2000, 1000, 2000}	1666.6
0.1	{2000, 1000, 2000, 1000}	1500

The scalar evaluation of the average of the salaries of the young people Y is then:

$$(1 - 0.4) \times 1500 + (0.4 - 0.1) \times 1666.66 + (0.1 - 0) \times 1500 = 1550.$$

The arithmetic on $\mathbb{Q}_f$ allows to evaluate an average function. Hence, the average A of a fuzzy bag S of values x_i is evaluated just as an extension on $\mathbb{Q}_f$ of a usual arithmetic weighted mean but considering fuzzy numbers of occurrences. It is defined as follows:

$$A = \left(\sum_{i=1}^{n} \Omega_S(x_i) \times x_i \right) \div \sum_{i=1}^{n} \Omega_S(x_i). \tag{40}$$

Due to the algebraic properties of $\mathbb{Q}_f$, it is worth noticing that the following property holds:

$$\sum_{i=1}^{n} \left[\Omega_S(x_i) \div \sum_{i=1}^{n} \Omega_S(x_i) \right] = 1.$$

Example 5.2 (*continued*). The collection of salaries of the young people Y is the fuzzy bag: $S = \langle 1/2000, 1/1000, 0.4/2000, 0.1/1000 \rangle$, which can also be represented using fuzzy cardinalities by $S = \{\{1/0, 1/1, 0.4/2\}*2000, \{1/0, 1/1, 0.1/2\}*1000\}$. The relative weights of the salaries 2000 et 1000 are respectively:

$$w_1 = \{1/1, 0.4/2\}^c \div [\{1/1, 0.4/2\}^c + \{1/1, 0.1/2\}^c] = \{1/1 \div 2, 0.4/2 \div 3, 0.1/1 \div 2\}^c,$$

$$w_2 = \{1/1, 0.1/2\}^c \div [\{1/1, 0.4/2\}^c + \{1/1, 0.1/2\}^c] = \{1/1 \div 2, 0.4/1 \div 3, 0.1/1 \div 2\}^c.$$

If we use an approach by α-cuts (cf. 4.2), we easily check that: $w_1 + w_2 = 1$. Because a integer n can be represented by the fuzzy number $\{1/n\}^c$, the average of salaries can be performed by

$$\begin{aligned} A &= (w_1 \times \{1/2000\}^c) + (w_2 \times \{1/1000\}^c) \\ &= \{1/2000 \div 2, 0.4/4000 \div 3, 0.1/2000 \div 2\}^c + (\{1/1000 \div 2, 0.4/1000 \div 3, 0.1/1000 \div 2\}^c \\ &= \{1/1500, 0.4/5000 \div 3, 0.1/1500\}^c. \end{aligned}$$

This fuzzy rational number A corresponds to an exact representation of the average of the salaries of the young people Y. From this exact evaluation, it is then possible to extract different estimates. A scalar estimate can be computed thanks to a Lebesgue integral:

$$E(A) = (1 - 0.4) \times 1500 + (0.4 - 0.1) \times 1666.66 + (0.1 - 0) \times 1500 = 1550.$$

This result is similar to the scalar value obtained via the previous approach originally proposed by Dubois and Prade which exploits a random set view of a fuzzy set.

5.3. Divisions and similarity measures

Inclusion grades and similarity measures between two fuzzy sets A and B are very numerous in the literature. Thus, there is a family of comparison indices based on the cardinalities of fuzzy sets, such as

$$\text{inclusion\_index}(A, B) = \frac{|A \cap B|}{|A|},$$

$$\text{equality\_index}(A, B) = \frac{|A \cap B|}{|A \cup B|},$$

$$\text{overlap\_index}(A, B) = \frac{|A \cap B|}{|A| \times |B|}.$$

The above definitions are generally based on a scalar evaluation of the cardinality of a fuzzy set F, defined by: $|F| = \sum_{x \in U} \mu_F(x)$. Such an evaluation (called Σ Count(F)) can be viewed as an approximation on $\mathbb{R}$ of the FGCount(F) function defined by Zadeh. Consequently, using Σ Counts the above indices are obtained by a calculus over approximations.

In Section 2.2, the fuzzy cardinality FGCount(F) is considered as an exact representation of the cardinality of F and it is interpreted as a fuzzy integer belonging to $\mathbb{N}_f$. Following the approach developed in this paper, the idea pushed forward in this subsection, is that, using the arithmetic on $\mathbb{Q}_f$, it is possible to naturally and straightforwardly perform exact evaluations of these indices. Then, in an ultimate step, from the resulting exact evaluations, it is possible to extract different estimates. For example, scalar estimates can be computed using a Lebesgue integral as it has been done in Section 5.2 for the average.

Example 5.3. Let A and B be fuzzy sets: $A=\{1/a, 0.6/b, 0.7/c, 0.7/d, 0.1/e\}$; $B=\{1/a, 0.6/b, 0.7/c, 0.7/e\}$. The scalar cardinalities of $A \cap B$ and A are

$$|A|_{\mathbb{R}} = 1 + 0.7 + 0.7 + 0.6 + 0.1 = 3.1,$$

$$|A \cap B|_{\mathbb{R}} = 1 + 0.7 + 0.6 + 0.1 = 2.4.$$

Based on these approximations, the usual inclusion index between A and B is

$$\text{inclusion\_index}(A, B) = \frac{2.4}{3.1} = 0.77.$$

In the framework of $\mathbb{Q}_\mathrm{f}$, we obtain

$$|A|_{\mathbb{Q}_\mathrm{f}} = \{1/1, 0.7/3, 0.6/4, 0.1/5\}^\mathrm{c},$$

$$|A \cap B|_{\mathbb{Q}_\mathrm{f}} = \{1/1, 0.7/2, 0.6/3, 0.1/4\}^\mathrm{c},$$

$$\text{inclusion\_index}(A, B)_{\mathbb{Q}_\mathrm{f}} = \{1/1, 0.7/2 \div 3, 0.6/3 \div 4, 0.5/4 \div 5\}^\mathrm{c}.$$

Based on this exact representation of the inclusion index on $\mathbb{Q}_\mathrm{f}$, we perform its approximation on $\mathbb{R}$ thanks to a Lebesgue integral:

$$\begin{aligned}\text{inclusion\_index}(A, B) = {} & (1 - 0.7) \times 1 + (0.7 - 0.6) \times 0.66 + (0.6 - 0.1) \times 0.75 \\ & +(0.1 - 0) \times 0.8 = 0.82.\end{aligned}$$

The relative cardinality of a fuzzy set A with respect a fuzzy set B can be interpreted as the proportion of elements of B that are in A. Possibilistic and probabilistic definitions of this notion are proposed by Delgado et al. [8]. These definitions are based on divisions by α-cuts ($|(A \cap B)_\alpha| \,/\, |(B)_\alpha|$), similarly to our approach. However, our interpretation of the result is specific. As explained in Section 2.2 for the representation of fuzzy integers, we claim that possibilistic, probabilistic or representation based on FGCount (a fuzzy rational is a couple of fuzzy natural integers) of a relative cardinality can be viewed as three equivalent representations of the *same* well-defined information corresponding to a fuzzy (gradual) rational quantity.

Finally, in the framework of $\mathbb{Q}_\mathrm{f}$, we notice that the previous indices can be easily generalized to compare two fuzzy bags because the cardinality of a fuzzy bag is a fuzzy integer (7).

6. Conclusion

This paper extends previous propositions related to fuzzy bags found in the following articles [22–25]. These works aim at defining the concept of fuzzy bag using a well-founded framework in which sets, fuzzy sets and bags can be viewed as particular cases of fuzzy bags. All of these structures are homogeneous and compatible because they are defined through a common mechanism: fuzzy cardinalities. Consequently, a small number of generic operators can be applied to these different collections. Using this approach, a fuzzy cardinality is interpreted as a gradual number which, taken as a whole, completely and exactly describes the cardinality of a fuzzy collection. This view of fuzzy numbers differs from the usual one, interpreting a fuzzy number as a possibility distribution describing the ill-known values of one variable, and has important consequences with regards to algebraic properties. Furthermore, we have shown that such a context offers powerful tools allowing the expression of flexible queries addressed to usual databases taking into account both preferences and quantities. However, it is worth mentioning that the considered concepts are general and can be applied to many other domains.

In this paper, this point of view is enlarged and we evoked how the set of natural integers ($\mathbb{N}_\mathrm{f}$) has been extended to the set of relative integers ($\mathbb{Z}_\mathrm{f}$). In this framework, the difference between two fuzzy integers is always exactly defined in terms of one equivalence class of pairs of fuzzy natural integers. It has been shown that each equivalence class can be identified by a unique canonical representative and can easily be manipulated using α-cuts. This approach has been pursued by extending $\mathbb{Z}_f$ to $\mathbb{Q}_f$ (the set of fuzzy rational numbers) where each number is defined in terms of one equivalence class of pairs of fuzzy relative integers also identified by a unique canonical representative. It is noteworthy that the set of fuzzy integers, with both addition and product, forms a ring and the rational numbers form the algebraic structure of a field. These new frameworks provide an arithmetic basis where difference or ratio between fuzzy quantities can be exactly evaluated. The obtained results can then be composed and used inside more complex calculations. Next, from an exact evaluation of an arithmetic expression on $\mathbb{Z}_f$ or $\mathbb{Q}_f$, it is possible to extract different approximations, on $\mathbb{N}_f$ or $\mathbb{R}$, for example, depending on users or applications needs.

In the future, complementary studies have to be carried out so as to define fuzzy order relations. Such comparisons between (fuzzy or crisp) quantities are essential in particular for dealing with flexible queries using absolute or relative fuzzy quantifiers such as: *find the best five companies in which the number of young employees is greater than the number of well-paid employees* or *find the best five companies in which most of the young employees are well-paid*. A preliminary study, which can be found in [26], shows that these fuzzy order relations rely on difference or division operators on $\mathbb{Q}_f$ and can be viewed as a generalization of fuzzy $\mathbb{R}$-implications. These investigations will be developed in future works.

Mathematicians in the past have introduced the notions of fuzzy real line [16] and subspaces of the real line [19]. We think that $\mathbb{Q}_f$ and the characterization of fuzzy order relations should open the door to the construction of $\mathbb{R}_f$ using a generalization of Dedekind cuts or Cauchy sequences.

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